Chapter 9
Grothendieck Toposes as Unifying 'Bridges': A Mathematical Morphogenesis

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Abstract We present some philosophical principles underlying the theory of topos-theoretic 'bridges', introduced by the author in 2010 and further developed and applied in the subsequent years.

Keywords Unification · 'Bridge' · Invariant · Sheaf · Topos · Site · Equivalence · Duality · Symmetry · Translation

9.1 Introduction

In this paper our aim is to expose some of the philosophical principles underlying our view of Grothendieck toposes as unifying 'bridges' between different mathematical contexts or theories. This view first emerged in Caramello (2010) and was further developed both theoretically (see Caramello 2017) and in relation to specific applications in different fields of Mathematics throughout the past years; see Caramello (2016a) for an overview of the main results obtained so far by applying this methodology. Thanks to these bridges, we can effectively relate – often in profound and unexpected ways – notions, properties and results of different mathematical theories that may well belong to seemingly distant fields and look disconnected at a first glance. In other words, these techniques enable us to multiply,

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in a sense, points of view on a given problem and to discover hidden relations between distinct mathematical contexts.

A clarification on terminology is needed. In this text, we shall widely use the term 'object' not (only) in the technical sense of category theory, but to refer to any entity, whether abstract or concrete, considered in the context of our methodology. So an 'object' could be a notion, a concept or also a concrete entity belonging to the 'real' world. Similarly, our use of the term 'construction' or 'building' of new objects from given ones should not be intended concretely, as it can refer to an abstract way for associating new objets with given ones (as it is indeed the case in the context of topos-theoretic 'bridges'), not necessarily in a constructive way. When we say 'topos', we always mean 'Grothendieck topos'.

Before proceeding to describe how the 'bridge' technique is implemented in the context of toposes, we shall clarify the sense in which we shall talk about unification and introduce the general concept of a 'bridge' object, as it has been inspired by these mathematical investigations. For examples of 'bridges' of non-mathematical nature, we refer the reader to Caramello (2016b, 2018), while an illustration of selected topos-theoretic 'bridges' is provided by Caramello (2016a).

Lastly, a disclaimer. While the general idea of a 'bridge object', as presented in Sect. 9.3, does not even require a mathematical background to be understood, our comments on its implementation in the context of toposes can hardly be appreciated without a basic knowledge of the language of category theory. As an introduction to this subject, we recommend the classical but still excellent book Mac Lane (1971) by S. Mac Lane. Readers interested in the philosophical and historical aspects of category theory may consult (Krömer, 2007; Marquis, 2010; Mazur, 2008).

9.2 What Does 'Unifying' Mean?

There are two different main significations to the concept of unification. One usually refers to a unifying framework as a general context subsuming a number of particular instances. So, for example, the concept of a category is unifying in the sense that concepts as different as that of a preorder or that of group can all be seen as particular cases of it. Similarly, the language of set theory (in one or another of its standard formalizations) is unifying in that one can ultimately express all mathematical concepts in terms of sets.

Still, one immediately realizes that this kind of unification, based on *generalization*, is *static* in the sense that being able to fit different objects in one context does not provide by itself a means for transferring knowledge between them. For example, knowing that the notions of preorder and of group are both particular cases of the notion of category does not give by itself a tool for transferring results about preorders to results about groups (or conversely), since such results are not necessarily specializations of general results about categories in these two contexts; in fact, the most interesting theorems about preorders or groups are *not* of this form, since they exploit in a non-trivial way the specific characteristics of the given objects, i.e. precisely the aspects of them that are lost in the generalization! This illustrates the fact that, in a generalization, the diversity of the single objects is not at all valorized, but rather 'forgotten'. In fact, any abstraction process consists in focusing the attention on a limited number of aspects of the given situation at a time, and drawing inferences exclusively on the basis of them, temporarily forgetting all the other ones (see also Sect. 9.11 below for more on abstraction in mathematics).

It is therefore natural to wonder if it is possible to achieve a more substantial form of unification which takes into account the diversity of the single objects in a significant way, rather than 'diluting' it in a generalization, and which is *dynamic* in the sense of allowing non-trivial transfers of information between the objects. One would want a unification that not only valorizes the diversity of the given objects but possibly *explains* its origin in relation to a unity lying somewhere else, as in a sort of *morphogenesis*.

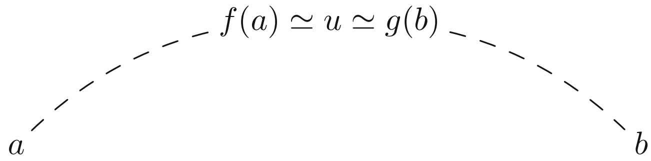
Our (positive) answer to these questions involves the notion of a *bridge object*. We shall discuss for simplicity the construction of 'bridges' across two given objects, even though the technique, as it will be clear, is applicable to arbitrarily big collections of objects.

9.3 The Idea of 'Bridge'

When we compare two objects with each other, we are interested in identifying the *invariants*, that is, the aspects that they have in common despite their diversity. These aspects can sometimes be identified in a concrete way, that is without the need to go out of a (relatively restricted) environmnent in which the objects lie. But most often, it is necessary to adopt novel points of view on the two objects capable of enlightening their (more or less hidden) invariants. This is what 'bridge' objects can achieve.

We can define a *bridge object* connecting two objects a and b as an object u which can be constructed from (or associated with – not necessarily in a concrete sense), from any of the two objects (independently from each other) and admits two different 'presentations' $f(a)$ and $g(b)$ (typically expressing the ways in which u is 'built' respectively from a and from b), in the sense that there is some kind of identification (in mathematics this is typically formalized as an equivalence relation) $\simeq$ both between u and $f(a)$ and between u and $g(b)$.

Such a 'bridge' object u allows us to build 'bridges' allowing *transfers of information* between a and b, as follows. For any property or notion applicable to u which is invariant with respect to the relation $\simeq$, one should attempt to 'translate' it into a property or notion applicable to the object a (resp. to the object b) by using the presentation $f(a)$ (resp. $g(b)$) of the bridge object u in terms of the object a (resp. b).



For instance, in the case of an invariant property P applicable to u, one looks for logical relationships of the kind '$f(a)$ satisfies P if and only if a satisfies a certain property P_a' and '$g(b)$ satisfies P if and only if b satisfies a certain property Q_b'; in such a situation, one gets a logical equivalence between P_a and Q_b, since both properties correspond to the same invariant property P of the bridge object, but understood from the points of view of the objects a and b via the presentations $f(a)$ and $g(b)$ of u. Notice that, since a and b can be objects of very different nature, the properties P_a and Q_b can be concretely very different, in spite of the fact that they represent different manifestations of a unique property, namely P, of the bridge object u.

Our notation $f(a)$ and $g(b)$ for the different presentations of the bridge object u is motivated by the mathematical applications, where these presentations of u are typically the result of applying some functions f and g respectively to a and to b. In general, though, $f(a)$ (resp. $g(b)$) may be any object associated with a (resp. b) by means of some kind of 'assignment', 'procedure' or 'construction' f (resp. g). In most cases, the passage from a (resp. b) to $f(a)$ (resp. $g(b)$) will involve a loss of information about a and b; that is, in general, one will not be able to 'reconstruct' (the whole of) a (resp. b) from $f(a)$ (resp. $g(b)$). Still, some essential information about a and b must be retained in this passage for non-trivial bridges to be established.

Bridge objects are in a sense universal invariants, since, almost tautologically, all the invariants considered on them 'factor' through them. In general, every bridge object supports infinitely many invariants (as any 'genuine' property of a bridge object is, essentially by definition, an invariant). Any invariant allows to transfer different information, behaving like a 'pair of glasses' capable of 'discerning' (or 'unveiling') hidden connections 'coded' in the equivalence between the two presentations of the bridge object. Of course, each of these transfers of information is partial, since each invariant embodies only *some* of the aspects that the two objects have in common. Ideally, in order to achieve a full transfer of information, one would need to consider all the possible invariants, and therefore all the possible bridge objects (and higher-order architectures involving them) connecting the two given objects.

The complexity of the 'unravelings' of properties of the bridge objects in terms of properties of its presentations (when they are at all feasible, in a non-tautological way) may vary enormously from case to case, depending on the sophistication of the invariant considered on u and on that of the constructions of the bridge object u from the two objects a and b. Still, the kind of unification realized by such a method is much more substantial than that achieved by a generalization. Indeed, the diversity of the two objects a and b is no longer 'forgotten', but becomes directly responsible for the different *forms* in which the invariants defined at the level of the bridge object manifest. So, we have a sort of *morphogenesis* which *explains* the origin of the diversity of different expressions of the same invariant.

Notice that in the 'bridge' technique, it is not the objects themselves to be unified, but their properties, or notions involving them. In other words, the unification takes place at an higher, more abstract level than that of the two objects. In fact, we have used the metaphor of the 'bridge' to underline the fact that we have two distinct levels, that of the objects to be investigated in relation to one another, and that of the bridge objects susceptible of connecting them.

It is important to bear in mind that a 'bridge' object may have a completely different nature from those of the objects which it relates to each other; the two levels do not in general collapse to a single one. The 'flat' bridges (i.e. those in which two levels can be identified) are the relatively uninteresting ones, since they correspond to the situations where the two objects can be directly connected to each other, and in which the unification boils down to 'homogeneisation'. In fact, in the context of theories with the same semantic content connected to each other by their common classifying topos (acting as a 'bridge' object between them), the pairs of theories that can be connected directly to each other are those which are bi-interpretable (in the sense of having equivalent syntactic categories), that is between which there exist 'dictionaries' (provided by the bi-interpretation) allowing to directly relate their syntaxes (cf. section 2.2 of Caramello 2017).

Any 'bridge' connects two levels, which can be thought of as the level of the 'contingent' (or 'concrete'), to which the objects to be unified belong to, and that of the universal (or 'abstract'), where bridge objects and the invariants defined on them lie.

The 'bridge' technique notably provides an approach to the problem of classifying invariants with respect to a given equivalence relation, and obtaining canonical representatives for the equivalences classes. Indeed, suppose that one wants to compare two objects a and b belonging to a set I on which an equivalence relation $\sim_I$ is defined. In such a situation, it is important to identify (and, possibly, classify) the properties of the objects of I that are *invariant* with respect to the relation $\sim_I$, since any such property will allow a transfer of information between a and b. Depending on the cases, this can be an approachable task or an hopelessly difficult one. In fact, a relationship between two given objects is in general an *abstract* entity, which lives in an ideal context which is generally different from the restricted, 'concrete' environment in which one typically considers the two objects (note that, even when the objects themselves are abstract, a relationship between them lies at a higher level of abstraction since it requires a higher degree of logical complexity to be formalized). It thus becomes of crucial importance to identify more 'concrete' (that is, 'easier to represent' or to investigate') entities which could act as 'bridges' connecting the two given objects, by representing in particular their common equivalence class in a form which is as 'concrete' (again, in the sense of ' manageable', or 'easily representable by the human mind') as possible. Think for instance of the complex plane, which is formally defined as the quotient $\mathbb{R}[x]/(x^2 + 1)$, but whose elements can be concretely represented as pairs of real numbers, or to the set $\mathbb{Q}$ of rational numbers, defined as a quotient of the product $\mathbb{Z} \times \mathbb{Z}^*$ (where $\mathbb{Z}^*$ denotes the set of non-zero integers), whose elements, which are equivalence classes, admit as canonical representatives the reduced fractions whose numerator (or denominator) has a fixed sign.

For this, the concept of *invariant construction* is relevant. We define an invariant construction $f : (I, \sim_I) \to (O, \sim_O)$ between sets I and O on which are defined equivalence relations $\sim_I$ and $\sim_O$, as a function $f : I \to O$ which respects the equivalence relations (i.e., such that whenever $x \sim_I y$, $f(x) \sim_O f(y)$). An invariant construction f is said to be *conservative* if it reflects the equivalence relations (i.e., whenever $f(x) \sim_O f(y)$, $x \sim_I y$). In such a situation, a bridge object connecting two objects $x, y \in I$ will be an object $u \in O$ such that $u \sim_O f(x)$ and $u \sim_O f(y)$. If f is a conservative invariant construction $(I, \sim_I) \to (O, \sim_O)$ then bridge objects in O notably represent equivalence classes of objects of I modulo the equivalence $\sim_I$.

Of course, the usefulness of such bridges greatly depends on whether it is more manageable to work with objects of type O than with objects of type I, or when the relation $\sim_O$ is more tractable than the relation $\sim_I$. Still, experience (both in mathematics and in other sciences) shows that, in the majority of situations, one needs, in order to effectively connect two objects of I, to move from the level of I to the level O of another object being able to serve as a 'bridge' across them (see also Sect. 9.11 for a discussion of this point in the context of the topos-theoretic 'bridge technique).

We anticipate that, in the context of our topos-theoretic 'bridges', the objects a and b to be investigated in relationship with each other will be specific mathematical contexts (represented as sites, theories or other objects from which toposes can be constructed), and $f(a)$ and $g(b)$ will be toposes attached to them which capture a 'common essence'. The 'bridge' technique can be notably applied in the context of theories classified by the same topos, for transferring information across them. Recall (Makkai and Reyes 1977) that any mathematical theory of a very general form (technically speaking, a geometric theory) admits a classifying topos, which, by definition, classifies its models in arbitrary toposes and thus embodies its semantic content (see also Marquis (2010) for an excellent conceptual introduction to categorical logic). Two theories are classified by the same topos (i.e. are *Morita-equivalent*) when, broadly speaking, they describe the same structures in their respective (possibly very different) languages. As we shall see in Sect. 9.6, the construction of the classifying topos defines a conservative invariant construction from the collection of geometric theories (endowed with the notion of Morita equivalence) to that of Grothendieck toposes (endowed with the notion of categorical equivalences of toposes), and classifying toposes can effectively act as 'bridge' objects across Morita-equivalent theories.

Examples of 'bridges' outside mathematics are discussed in Caramello (2016b, 2018).

9.4 Sheaves, or the Passage from the Local to the Global

Grothendieck toposes (Artin et al. 1972) are, by definition, categories (equivalent to a category) of sheaves (of sets) on a site. The notion of site arises from an abstraction of the notion of covering of an open set by a family of open subsets in a given topological space, and represents the most general categorical context for defining sheaves. The notion of sheaf on a topological space was introduced by J. Leray: a sheaf on a topological space X is a way of assigning to each open set U of X a set $\mathcal{F}(U)$ and to each inclusion between open sets $V \subseteq U$ a map $\rho_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ in such a way that $\rho_{U,U} = 1_{\mathcal{F}(U)}$ for each U and $\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U}$ for each $W \subseteq V \subseteq U$ (these are the conditions that define the notion of *presheaf* on X) plus the requirement that, for any open covering of U by a family $\{U_i \mid i \in I\}$ of open subsets $U_i \subseteq U$, giving an element of $\mathcal{F}(U)$ corresponds precisely to giving a family $\{x_i \in \mathcal{F}(U_i) \mid i \in I\}$ of elements of the $\mathcal{F}(U_i)$'s which is compatible in the sense that $\rho_{Z,U_i}(x_i) = \rho_{Z,U_j}(x_j)$ for each $i, j \in I$ and any open set Z contained both in U_i and in U_j.

The canonical example of a sheaf on a topological space X is that of continuous real-valued functions, which assigns to each open set of X the set of continuous real-valued functions on U and to each inclusion $V \subseteq U$ between open sets the operation of restriction of such functions on U to V.

Categorically, a presheaf is simply a functor from the opposite of the category $\mathcal{O}(X)$ of open sets of X (whose objects are the open sets of X and whose arrows are the inclusions between them) to the category of sets.

Sheaves on a topological space X can be defined as presheaves satisfying the above gluing condition, which can be expressed categorically entirely in terms of the category $\mathcal{O}(X)$ and of the notion of covering family $\{U_i \hookrightarrow U \mid i \in I\}$ in this category.

A (small) site is a pair $(\mathcal{C}, J)$ consisting of a (small) category $\mathcal{C}$ and a so-called *Grothendieck topology* J on it, which specifies a notion of 'covering family' of arrows in $\mathcal{C}$, with respect to which one can formulate a sheaf condition.

A presheaf on a category $\mathcal{C}$ is simply a contravariant functor from $\mathcal{C}$ to the category **Set** of sets. Given a Grothendieck topology J on $\mathcal{C}$, a presheaf P on $\mathcal{C}$ is said to be a J-*sheaf* if it satisfies the glueing condition with respect to every compatible family of elements of P indexed by a J-covering family; see Mac Lane and Moerdijk (1992) for the details. The category of J-sheaves on $\mathcal{C}$ and natural transformations between them is denoted by $\mathbf{Sh}(\mathcal{C}, J)$. A Grothendieck topos is any category $\mathcal{E}$ equivalent to the category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on a small site $(\mathcal{C}, J)$.

The notion of sheaf expresses a very robust and harmonious relationship between the local and the global. It formalizes the process by which one defines a global entity by specifying its local behaviour on objects covering its domain. For instance, one can define a continuous real-valued function on an open set U of a topological space X by pasting together continuous real-valued functions $f_i : U_i \to \mathbb{R}$ defined on sub-open sets U_i covering U which are compatible with each other (in the sense that they agree on the intersections of the U_i's).

Notice that every set of local data which comes from a global entity by a process of 'instantiation' satisfies some coherence conditions; for example, given a continuous function f defined on an open set U of a topological space, if we define f_V as the restriction of f to an open subset $V \subseteq U$ and f_W as the restriction of f to an open subset $W \subseteq U$, then we have the coherence relation $f_V|_Z = f_W|_Z$ for any open subset $Z \subseteq V \cap W$. The definition of sheaf requires the converse also to hold: any set of compatible local data should uniquely determine a globally-defined *datum*.

The theory of descent data is another (higher-categorical) illustration of the same principle of defining global entities by gluing compatible sets of local data.

Although sheaves might appear very abstract at first sight, they are actually real in a very strong sense. What is reality if not a sheaf of coherent perceptions? Note that we tend to call 'real' anything which is 'independent from' (in the sense of 'invariant with respect to') the perceptions that we might have of it. The reason why we believe in the existence of reality (if we do) is that we continuosly experience coherence relations existing between the perceptions of different individuals or measuring instruments; it is therefore scientifically reasonable (as a minimalist explanation) to suppose the existence of something which would 'generate' (by its mere existence) all these perceptions and hence which would be ontologically responsible for the coherence relations between them.

9.5 Grothendieck Toposes

A general sheaf is by itself a rather rich entity since is specifies not only a fixed, global *datum* but a whole collection of local data which are compatible with each other. Considering all sheaves on a given site to form a category, namely a Grothendieck topos, adds even more coherence relations arising from the 'interaction' between different sheaves. It is thus clear that a Grothendieck topos is an extremely rich entity. Indeed, any topos is a veritable mathematical universe within which one can do mathematics and in particular consider models of arbitrary first-order (and even higher-order) theories.

As a mathematical universe, every topos has its own internal logic, which in general is not classical but intuitionistic and with multiple truth values, reflecting the fact that the notion of truth accommodated by sheaves is local and variable (according to the domain of a generalized element), rather than global and fixed. As far as its internal structure is concerned, a Grothendieck topos satisfies all the completeness properties one might hope for: every universal problem (expressible in terms of existence of limits or colimits) has a solution in a topos, because of its categorical completeness and cocompleteness. Moreover, any topos has exponentials (which are the categorical analogue of the sets of functions from one given set to another), a subobject classifier, which encodes much of its internal logic, separating sets and coseparating sets.

These categorical properties have a number of remarkable consequences: every functor between toposes which preserves all limits (resp. colimits) has a left (resp.

right) adjoint; in particular, every covariant (resp. contravariant) functor from a Grothendieck topos to the category of sets which preserves limits (resp. which sends colimits to limits) is representable. All of this shows that toposes are ideal concepts to be used for building unifying 'bridges' across different mathematical contexts since they 'accommodate' (in the sense of being natural homes for) many objects arising as the solution of universal problems; a typical example is the notion of universal model of a geometric theory within its classifying topos. Moreover, this very rich categorical structure is responsible for the fact that toposes are full of symmetries or, in other words, that they naturally support a great number of invariants.

Grothendieck toposes are also stable with respect to many significant operations that one might want to perform on them; the 2-category of toposes is itself very rich. In particular, the theory of Grothendieck toposes supports relativisation techniques, since the topos of sheaves on a site internal (or relative) to a base Grothendieck topos is again a Grothendieck topos. Being able to change the base topos according to one's needs and to switch from an 'internal' to an 'external' point of view when dealing with toposes in relation to one another is a classical technique that adds further power to the theory (much as Grothendieck's relativisation techniques have played a key role in his refoundation of algebraic geometry in the language of schemes) and naturally leads to the discovery of a great number of different presentations for the same topos.

In a sense, the ontology of toposes is very large. Every mathematical theory, even a contradictory one, finds its home in the context of toposes (note that this is not true in the restricted context of sets, where a contradictory theory does not have any models); in fact, a geometric theory is contradictory if and only if its classifying topos is trivial (in the sense that it is the topos **1** having one object and one arrow on it). Note that, whilst being trivial, this topos does indeed contain a model of the theory, namely its universal model. This is very relevant both from a conceptual and a technical viewpoint; indeed, not having to worry whether something exists allows for a much greater technical flexibility. The problem is no longer whether the object we would like to construct exists or not; in the world of toposes, in a sense, all problems (of a specified but very general kind – think for instance of the existence of limits or colimits for small diagrams) admit a solution, so the 'absolute' problem of the existence of a desired entity gets reduced to a 'relational' problem, that of whether we can transport the 'universal', topos-theoretic solution to a set-theoretic or more concrete structure suitable for our needs (think, as an example, of the topos-theoretic construction of forcing models for set theory). Notice that this very large ontology manifests itself both at the level of the individual objects of toposes, namely sheaves and at the global level of the entire universe of sheaves on a given site, that is, at the level of a given topos, in addition to the level of the (very big) 2-category of toposes.

The 'completeness' of the world of toposes is also reflected in the fact, already hinted at above, that all the functors between toposes that satisfy the necessary conditions for being representable (resp. for admitting a left or right adjoint) *are* indeed representable (resp. *do admit* such a left or right adjoint).

Another very relevant aspect of toposes is their amenability to computation. Any Grothendieck topos is, in a sense, a mathematical environment without ‘holes’, by virtue of the completeness properties it enjoys; this makes it very convenient for calculations, since one can exploit all its internal symmetries for carrying them smoothly and effectively, never having to worry whether the result of this or that categorical operation exists or not. On the other hand, one does not go far astray by computing in a topos, since, following the ‘bridge’ philosophy, one can always interpret the results of these computations in terms of relevant presentations for the given topos.

9.6 The Yoneda Paradigm

Recall that a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a way of assigning to each object of $\mathcal{C}$ an object of $\mathcal{D}$ and to each arrow of $\mathcal{C}$ an arrow of $\mathcal{D}$ so as to respect identity arrows and the operations of domain, codomain and composition of arrows.

We can think of a functor as a carrier of information which is indexed by the objects and arrows of the domain category. In particular, a presheaf P on a category $\mathcal{C}$, which by definition is a contravariant functor from $\mathcal{C}$ to the category **Set** of sets, sends any object c of $\mathcal{C}$ to a set $P(c)$ and any arrow $f : d \to c$ in $\mathcal{C}$ to a map $P(f) : P(c) \to P(d)$. As such, a presheaf is in general a carrier of a significant amount of information. Any object c_0 of the category $\mathcal{C}$ determines a presheaf on $\mathcal{C}$, denoted by $\text{Hom}_{\mathcal{C}}(-, c_0)$, which sends an object c of $\mathcal{C}$ to the collection $\text{Hom}_{\mathcal{C}}(c, c_0)$ of arrows in $\mathcal{C}$ from c to c_0 and sends an arrow $f : d \to c$ in $\mathcal{C}$ to the function $\text{Hom}_{\mathcal{C}}(c, c_0) \to \text{Hom}_{\mathcal{C}}(d, c_0)$ between these home sets given by composition with f on the right.

Recall that a presheaf P is said to be *representable* if it is (up to isomorphism) of the form $\text{Hom}_{\mathcal{C}}(-, c_0)$ for some object c_0 of $\mathcal{C}$. This means that there is an element $x_0 \in P(c_0)$ which ‘generates’ all the elements of P in the sense that for any object c of $\mathcal{C}$ and any element $x \in P(c)$ there is a unique arrow $c \to c_0$ such that $P(f) : P(c_0) \to P(c)$ sends x_0 to x. When a functor is representable, this means that all the information carried out by it is actually concentrated in a single object, namely the pair (c_0, x_0) representing it. If the functor carries a lot of information, proving its representability is a significant result, since it shows that the functor has a sort of ‘center of symmetry’, given by its representing pair, which generates by ‘deformation’ all its elements associated with arbitrary objects of the category.

The assignment $c_0 \mapsto \text{Hom}_{\mathcal{C}}(-, c_0)$ can actually be made into a functor

$$y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}],$$

called the *Yoneda embedding* of $\mathcal{C}$ into the category $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ of presheaves on $\mathcal{C}$. This functor is full and faithful; in particular, it reflects isomorphisms. This shows that an object c can be identified with the corresponding representable functor $\text{Hom}_{\mathcal{C}}(-, c)$, whose elements are the arrows from arbitrary objects of $\mathcal{C}$ to c. These arrows are called the *generalized elements* of c; this terminology is justified by the fact that, in the category **Set** of sets, any element of a set c can be identified with an arrow $1 \to c$ in **Set**, where 1 is the singleton set $\{*\}$. A generalized element of an object c, as an arrow to c, thus defines a point of view on c, or better a 'direction of observation' of c within the category $\mathcal{C}$. We can thus interpret the above result by saying that an object can be identified with its generalized elements, that is, broadly speaking, with the collection of points of view that we can have on it. We call this the *Yoneda paradigm*.

Notice that, in this context as well, there are coherence relations between the points of view that one can have on a given object: for instance, any arrow $b \to a$ in $\mathcal{C}$ canonically induces a way of mapping the generalized elements of c defined on b to the generalized elements of c defined on a. These notions suggest that, in general, whenever one experiences coherence relations, one should look for the 'source' of them, possibly in the form of a representing pair for a functor encoding such relations (see also Sect. 9.7 below). As we argued in Sect. 9.4, the language of sheaves is particularly apt to formalize local-global coherence relations. Grothendieck had the key idea of considering *all* sheaves on a given site to form his toposes, supported by the conviction that, since each of them behaves as a sort of " meter[1]" of the site, it is all the more powerful to consider a measure instrument not in an isolated way but in connection with all the other measure instruments that one might want to dispose of. This in fact leads to a whole universe of coherence relations, namely a topos.

The identification of the objects c of a category $\mathcal{C}$ with their functors $\mathrm{Hom}_{\mathcal{C}}(-, c)$ of generalized elements realized by the Yoneda embedding is very relevant also from a technical viewpoint, since it allows to understand constructions internal to the category $\mathcal{C}$ in set-theoretic terms. For example, since the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ preserves and reflects all limits, limits in $\mathcal{C}$ can be understood in terms of limits in the corresponding presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, which are in turn calculated componentwise in terms of the relevant limits in **Set**.

The Yoneda lemma has a beautiful incarnation in the context of toposes, which provides a further illustration of their natural symmetries and completeness properties. Every Grothendieck topos $\mathcal{E}$ can be endowed with a Grothendieck topology $J_{\mathcal{E}}^{\mathrm{can}}$, called the canonical one, whose covering sieves are those which contain small epimorphic families; this is the largest Grothendieck topology on $\mathcal{E}$ for which all the representable functors are sheaves. Now, the Yoneda embedding $y_{\mathcal{E}} : \mathcal{E} \to [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$ yields an equivalence

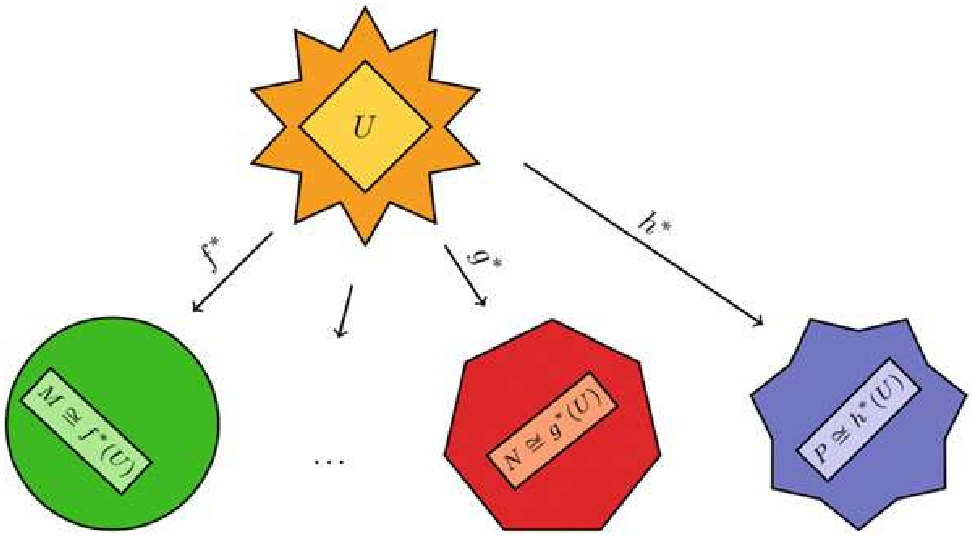
$$\mathcal{E} \simeq \mathbf{Sh}(\mathcal{E}, J_{\mathcal{E}}^{\mathrm{can}})$$

between $\mathcal{E}$ and the category of sheaves on this (large in general, but always small-generated) site. This result can be profitably applied not only for understanding

[1] See section 2.12 of his work *Récoltes et Semailles*.

limits in $\mathcal{E}$, but also for describing non-trivial constructions in $\mathcal{E}$, such as colimits, in terms of generalized elements of objects of $\mathcal{E}$.

Finally, we remark that the theme of representability plays a key role in connection with the topos-theoretic 'bridge' technique. Indeed, the classifying topos of a geometric theory can be defined as a representing object for the (pseudo)functor of models of the theory; its generalized elements (within the category of Grothendieck toposes) are the categories of models of the theory in arbitrary toposes. The Yoneda paradigm thus tells us that a Grothendieck topos can be identified with the (pseudofunctor of) structures that it classifies. In particular, it contains a distinguished model of the theory, called its *universal model*, which 'generates' all the models of the theory in arbitrary toposes:



Classifying topos

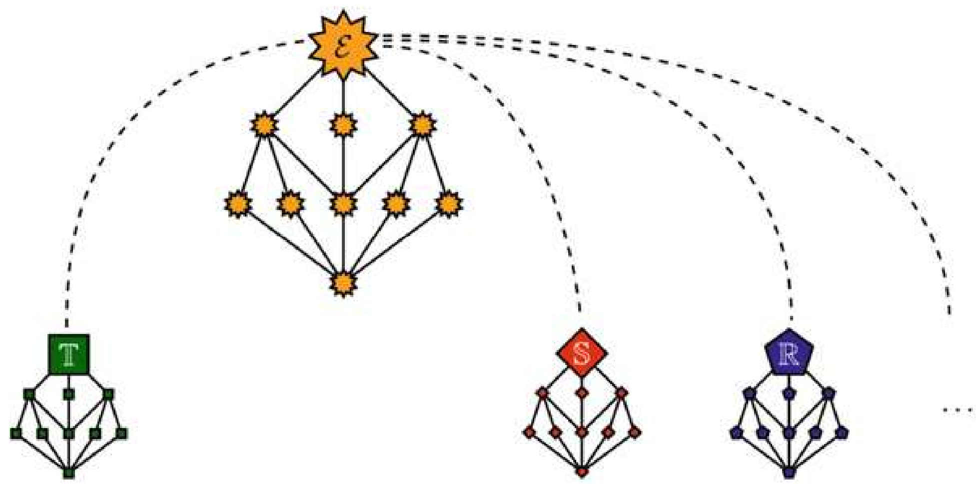
In the picture, the big coloured shapes represent different toposes while the inner lighter shapes represent models of a given theory inside them; in particular, the dark yellow star represents the classifying topos of the theory and the light diamond represents 'the' universal model of the theory inside it. The classifying topos thus resembles to a 'sun' generating shadows in all directions; when we look at particular models of the theory in toposes, we are just contemplating deformations of this universal model by means of structure-preserving functors (technically speaking, inverse images of geometric morphisms of toposes), a bit as if we were in Plato's cave.

By definition of classifying topos, two theories are Morita-equivalent (i.e. they have equivalent classifying toposes) if and only if they have equivalent categories of models in any topos, naturally in the topos, that is, broadly speaking, if they describe, each in its own language, the same structures, or, in other words, if they have the same mathematical content (embodied by the common classifying topos). Notice that the classifying topos construction constitutes a conservative invariant construction (in the sense of Sect. 9.3) from the world of geometric theories endowed with the relation of Morita equivalence to the world of Grothendieck toposes endowed with the relation of categorical equivalence; one can then understand, given the fact that the notion of categorical equivalence is much more technically tractable than Morita equivalence (for instance, one can easily see whether a property is a categorical invariant, and introduce infinitely many new categorical invariants without any effort), the crucial role that classifying toposes can play as 'bridges' across Morita-equivalent theories.

9.7 Generation from a Source

There are different senses in which one can understand the idea of 'generation from a source' which we hinted at in Sect. 9.4. We shall be concerned in particular with two of them. The first is the Yoneda paradigm of representable functors, discussed in Sect. 9.6; the second, which is more subtle as it lies at a higher level of abstraction, is the relationship between a bridge object and its different 'representations'. This duality is akin to that between a theme and a number of variations on it: anything which happens at the level of the bridge object will have 'ramifications' in the context of all its representations. Such 'ramifications' will then entertain coherence relations between them just as different 'variations' on the same 'theme' lying at the level of the bridge object, which then appears as ontologically 'responsible' for the existence of such relations.

For instance, the following picture represents the lattice structure on the collection of the subtoposes of a topos $\mathcal{E}$ inducing lattice structures on the collection of 'quotients' (i.e. geometric theory extensions over the same signature) of geometric theories $\mathbb{T}$, $\mathbb{S}$, $\mathbb{R}$ classified by it.



Lattices of theories

Notice that there are infinitely many theories classified by the same topos, which may belong to different areas of mathematics. So, if one ignores toposes, it would be difficult to realize that a particular structure we are dealing with in a specific mathematical context has in fact a counterpart in another field of mathematics just because this structure is induced by a universal structure lying at the topos-theoretic level. It would therefore be of great usefulness for the 'working mathematician' to realize about which of the construction (s)he deals with can be lifted at the topos-theoretic level, so that alternative versions of them can be obtained by switching to different representations of the same topos. Indeed, a topos is an object that, by embodying, in a sense, the collection of all points of view on a given topic, represents a crossroads between different mathematical paths, a place where different perspectives and languages converge mirroring one into another.

By making a topos-theoretic analysis of the concepts one works with, one is also able to understand whether the notions one is dealing with are sufficiently *robust* or *modular* (in the sense that they correspond to topos-theoretic invariants), in which case they admit infinitely many reformulations in different contexts providing multiple points of view on the given topic, or whether instead they are ad hoc, *concrete* notions that perhaps serve a very specific purpose but which occupy a relatively marginal place in the mathematical landscape.

In our context, there are also two main meanings that we can give to the expression 'point of view'. As argued in Sect. 9.6, we can think of a generalized element of an object in a category as a point of view on it. But it is also natural to think of a presentation of a bridge object as a point of view on it. In fact, in the context of classifying toposes, their different presentations correspond to different theories classified by them, which indeed provide, each with its own language, different points of view on these toposes.

When we talk about *morphogenesis* (see Sect. 9.3), we refer to the fact that different invariants on a bridge object may manifest themselves in different ways in the context of different presentations of that object. We can interpret this by saying that every form that exists abstractly at the level of a bridge object differentiates giving rise to different forms in the context of different presentations of that object. It is this process of 'differentiation from the unity' that we call 'morphogenesis'. As an example of the significantly different ways in which even basic invariants of toposes manifest in the context of different sites, consider the property of a topos to be bivalent: in the context of a trivial site $(\mathcal{C}, T)$, it corresponds to the property of $\mathcal{C}$ to be strongly connected (in the sense that for any objects a and b of $\mathcal{C}$, there is both an arrow $a \to b$ and an arrow $b \to a$), in the context of an atomic site $(\mathcal{C}, J_{\mathrm{at}})$ it corresponds to the property of $\mathcal{C}$ to be non-empty and to satisfy the joint embedding property (that is, the property that any two objects of $\mathcal{C}$ admit an arrow from a third one), and in the context of the classifying topos of a geometric theory $\mathbb{T}$ (represented as the category of sheaves on its syntactic site) it corresponds to the property that $\mathbb{T}$ be geometrically complete (in the sense that every geometric assertion in the language of the theory is either provably true or provably false in it, but not both). As another example, the property of a topos to satisfy De Morgan's law manifests in the context of a topos of the form $[\mathcal{C}, \mathbf{Set}]$ as the amalgamation property on $\mathcal{C}$ (i.e. the property that every pair of arrows with common domain can be completed to a commutative square), while in the context of the topos of sheaves $\mathbf{Sh}(X)$ on a topological space it corresponds to the property of X to be extremally disconnected (in the sense that the closure of any open set is open). As yet another example, take the property of a topos to be Boolean; in the context of a presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ it corresponds to the property of $\mathcal{C}$ to be a groupoid, while in the context of the topos of sheaves $\mathbf{Sh}(X)$ on a topological space it corresponds to the property of X to be almost discrete. Notice that these are relatively simple invariants of toposes that are very easily calculated; still, their different manifestations in the context of different presentations are very surprising (without the point of view of toposes, it would have been hard to imagine that they could be related to each other), which gives an idea of the mathematical morphogenesis induced by topos-theoretic invariants (which of course will be much greater than in the above examples in the case of more sophisticated invariants).

It is important to remark that, whilst 'concretely' very different, all the manifestations of a given invariant in the context of different presentations are *compatible* with each other and actually entertain coherence relations ultimately resulting from the fact that they all come from the same source. Most strikingly, this morphogenesis is entirely determined by the structural relationship between a topos (or, more generally, a bridge object) and its different presentations.

As an example of a 'bridge' obtained by using the above-mentioned invariants, we mention our topos-theoretic interpretation (Caramello 2014) of Fraïssé's theorem in Model Theory, where the classifying toposes of the theories $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models (where $\mathbb{T}$ is a geometric theory classified by a presheaf topos whose category f.p.$\mathbb{T}$-mod($\mathbf{Set}$) of its finitely presentable models satisfies the amalgamation property) are presented, on the one hand, as the categories $\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}})$ of sheaves on f.p.$\mathbb{T}$-mod$(\mathbf{Set})^{\text{op}}$ with respect to the atomic topology and, on the other hand, as the categories $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$ on the geometric syntactic site of $\mathbb{T}'$. Transferring the invariant property of being bivalent across these two presentations yields the following result: $\mathbb{T}'$ is complete if and only if f.p.$\mathbb{T}$-mod($\mathbf{Set}$) is non-empty and satisfies the joint embedding property. Notice that completeness is in general a hard-to-establish property of theories, while the joint embedding property, at least in a great number of situations, is a much more 'concrete' and easier property to investigate.

9.8 Sites and Toposes, or the Contingent and the Universal

The key element on which the 'bridge' technique is based is the fundamental ambiguity consisting in the fact that any given topos has infinitely many different presentations. Topos-theoretic invariants can thus be used for generating 'bridges' connecting such presentations. As in any 'bridge', we have two levels: that of the contingent, represented in this case by sites, axiomatic presentations of theories or other objects by means of which toposes can be presented, and that of toposes, which is the abstract level where invariants naturally live.

Every topos-theoretic invariant generates a veritable mathematical morphogenesis resulting from its expression in terms of different presentations of toposes, which often gives rise to connections between 'concrete' properties or notions that are completely different and apparently unrelated from each other.

The mathematical exploration is therefore in a sense 'reversed' with respect to the more classical, 'bottom-up' approaches since it is guided by the equivalences between different presentations of the same topos and by topos-theoretic invariants, from which one proceeds to extract concrete information on the theories or contexts that one wishes to study.

Toposes can be thought of as 'stars' that enlighten mathematical reality (cf. Sect. 9.6), as universal standpoints on theories which naturally unveil their symmetries. A site, or, more generally, an object from which a topos can be built, is, in a sense, a point of view on that topos. Any site $(\mathcal{C}, J)$ can be 'mapped' to the corresponding topos by means of the canonical functor

$$l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$$

to the topos of sheaves on it. Interestingly, there are infinitely many 'intermediate' sites between it and the associated topos (which can be considered itself a site, by equipping it with the canonical topology C), obtained by equipping any full subcategory of $\mathbf{Sh}(\mathcal{C}, J)$ containing the image of $\mathcal{C}$ with the Grothendieck topology induced by C. These sites can be viewed as different 'scales' of observation for phenomena formalized as topos-theoretic invariants; the way in which such invariants express in terms of them would then account for the existence of multiple descriptions of invariant laws at different scales. Having different and apparently incompatible descriptions for a given 'physical' content at different scales is a fundamental problem that physicists face; a topos-theoretic analysis of this kind of problems could therefore be highly beneficial.

A simple example illustrating this last remark is provided by the construction of the Alexandrov space $\mathcal{A}_{\mathcal{P}}$ associated with a preorder $\mathcal{P}$. Recall that $\mathcal{A}_{\mathcal{P}}$ is the topological space whose underlying set is $\mathcal{P}$ and whose open sets are the upper sets with respect to the preorder relation, that is the subsets U of $\mathcal{P}$ such that whenever $a \leq b$ in $\mathcal{P}$, $a \in U$ implies $b \in U$. Now, it is easy to see that we have a canonical equivalence

$$[\mathcal{P}, \mathbf{Set}] \simeq \mathbf{Sh}(\mathcal{A}_P) \ .$$

The first presentation $[\mathcal{P}, \mathbf{Set}]$ of this topos is in a sense 'combinatorial', while the second is topological; indeed, the objects of the first site are the elements of $\mathcal{P}$, considered as objects of a category and hence deprived of internal structure (as if they were elementary particles), while the objects of the canonical site presenting $\mathbf{Sh}(\mathcal{A}_P)$ are open sets of $\mathcal{P}$, which instead have a rich internal structure supporting a geometrical intuition. Accordingly, when a given topos-theoretic invariant is studied from the point of view of these two presentations, in the first case one obtains a combinatorial formulation of it, while in the second a topological formulation. Take for instance the property of a topos to be De Morgan; as we saw in Sect. 9.7, this reformulates in terms of $\mathcal{P}$ as the amalgamation property on it (that is, the property that for any elements $a, b, c \in \mathcal{P}$ such that $a \leq b$ and $a \leq c$ there is $d \in \mathcal{P}$ such that $b \leq d$ and $c \leq d$) and in terms of $\mathcal{A}_{\mathcal{P}}$ as the property of this space to be extremally disconnected.

Let us further elaborate more generally on this duality between the contingent and the universal.

Every language or point of view is partial and incomplete (i.e. full of 'holes'), and it is only through the integration of all points of view that we can capture the essence of things (cf. Sects. 9.7 and 9.6).

There is no universal language that would be better (in an absolute sense) than all the others; every point of view highlights certain aspects and hides others and can be more convenient than another in relation to a certain goal. Universality should thus be researched not at the level of languages (or 'points of view') but at the level of the 'ideal' objects on which invariants are defined.

It is therefore necessary to reason at two levels, that of the invariants (and the 'bridge' objects on which they are defined) and that of their manifestations in the context of 'concrete' situations, and to study the duality between these two levels, a duality that can be thought of as the one between a 'sense' and the different ways to express it. These two levels are independent from each other and important each in its own right; as observed in Sect. 9.3, confusing them makes unification collapse to 'homogenisation'.

9.9 Invariant-Based Translations

The 'bridge' technique is a methodology for translating notions, ideas and results across different mathematical contexts. It is important to realize that, in general, such translations are *not* literal, since they are determined by the expression of topos-theoretic invariants in terms of different presentations of toposes, rather than by the use of a 'dictionary'. In fact, as already remarked in Sect. 9.3, 'dictionaries', or direct ways of relating two given objects with each other, exist only in a minority of cases, which in fact are relatively uninteresting since the resulting translations do not essentially change the syntactical shape in which the information is presented and hence do not really generate novel points of view on it; these are precisely the situations where unification collapses to homogenisation.

In fact, even in Linguistics, a good translation is most often a non-literal one; such a translation should be based on a preliminary identification of the invariants, that is of the aspects of the text which one wants to remain unchanged (i.e. preserved) in the transition process from one language to the other. In the case of translations between natural languages, the most obvious invariant is meaning, but there are others too: for instance, one might also, or in alternative, want to preserve a particular type of metre or musicality, especially when translating a poem. Anyway, whatever the invariants, what matters is to let them guide the translation, that is play the role of bridge objects determining the translation by virtue of how they express in the two different languages. The same happens with the mathematical translations based on topos-theoretic ‘bridges’: in this case the objects to be related are mathematical contexts or theories and the bridge objects are toposes associated to them, on which an infinite number of invariants are defined.

9.10 Symmetries by Completion

As a matter of fact, the more one enlarges the mathematical environment where one works, the higher is the number of internal symmetries that one generates. Think for example of number systems; the development of mathematics has gone progressively in the direction of extending them in order to find solutions to certain problems (such as finding inverses to certain naturally defined operations such as addition, multiplication, taking powers etc.). For instance, from the set of natural numbers one has constructed the integers by formally adding negative numbers: by doing this, a fundamental symmetry with respect to the zero has appeared. Similarly, by passing from the real line to the complex plane in order to notably find a solution to the equation $x^2 + 1 = 0$, one has found a much more ‘symmetric’ mathematical environment, as witnessed in particular by the fundamental theorem of algebra (which provides a perfect symmetry between the degree of a polynomial in one variable and the number of its roots counted with their multiplicities, and which has no natural analogue in the restricted context of the real line).

On the other hand, to relate different languages or points of view to each other it is necessary to *complete* them to objects that realize *explicitly* the *implicit* hidden in each of them and which therefore can act as bridge objects connecting them. Indeed, it is at the level of these completed objects that invariants, that is, symmetries, manifest, and that we can understand the relationships between our given objects thanks to the ‘bridges’ induced by the invariants.

We can see all these principles incarnated by the use of toposes as ‘bridges’. As the complex plane $\mathbb{C}$ is obtained from the real line by means of a formal construction (namely, $\mathbb{R}[x]/(x^2 + 1)$) consisting in the addition of certain ‘imaginaries’, so the topos $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on a site $(\mathcal{C}, J)$ represents a completion of $\mathcal{C}$ by the addition of imaginaries (indeed, any object of $\mathbf{Sh}(\mathcal{C}, J)$ can be canonically expressed as a ‘definable’ quotient of a coproduct of objects coming from $\mathcal{C}$). Also, the classifying topos of a theory is constructed by means of a completion process (of the theory itself), with respect, in a sense, to all the concepts that she is potentially capable to express. Thanks to the ‘bridge’ technique, different theories that describe the same mathematical content are put in relation with each other as if they were fragments of a single object, partial languages that complement each other by mirroring each into one another within the totality of points of view embodied by the classifying topos.

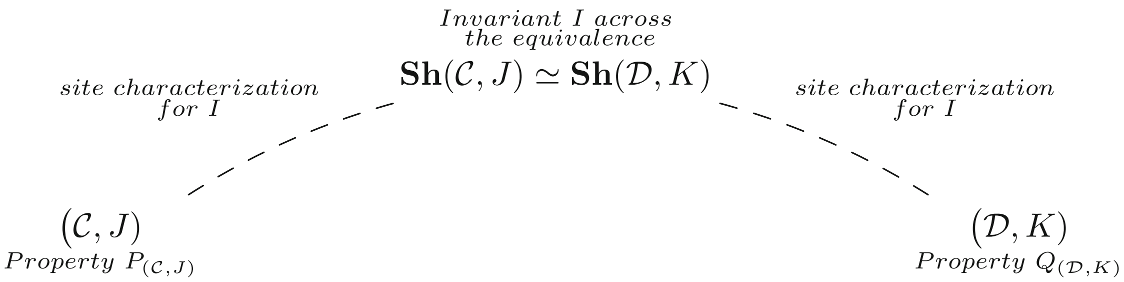
In fact, a translation should not be thought of as a way for relating entities that are necessarily very different from each other, but as a process of discovery of new potential implicit in a certain point of view or language. Every language, in an attempt to express a reality that is much richer, can be compared to a sketch drawn by an artist. In looking at it, our brain operates a sort of automatic completion of it which allows us to understand its implicit meaning. The transition from a linguistic expression to its meaning can thus be thought of as a kind of completion.

9.11 The ‘Bridge’ Technique in Topos Theory

As anticipated in the previous sections, the construction of a topos-theoretic ‘bridge’ involves first of all the identification of an equivalence between two different presentations for the same topos; this will form the ‘deck’ of the bridge, while the different objects presenting the toposes will constitute the extremes of the bridge (to be related with each other). Of course, one can also consider, as ‘decks’ of bridges, more general kinds of relationships between toposes that are not equivalences, but, whilst invariants under equivalences of categories are easily generated and identified, there are many less invariants available in the case of a relation that is not an equivalence. So, even if one starts with two toposes between which there is a relation that is not an equivalence, it is often convenient to try to modify the toposes involved by suitable operations in order to obtain an equivalence of toposes and then apply the ‘bridge’ technique to the latter.

Next, one considers a topos-theoretic invariant, and tries to ‘unravel’ it both in terms of the first presentation and in terms of the second (in a non-tautological way, that is obtaining genuine, ‘concrete’ expressions for it in the ‘languages’ of the two presentations); provided that this is feasible, these characterizations will give the two ‘arches’ of our bridge.

For example, a ‘bridge’ between different sites of presentations for the same topos induced by an invariant property of toposes, has the following form:



Here the properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$ of the sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, shown by the ‘bridge’ to be equivalent, are *unified* as different manifestations, in the context of the sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, of the same invariant I lying at the topos-theoretic level.

In practice, the choice of the invariant(s) will depend on the kind of information that one is interested to extract from the equivalence of toposes; indeed, the invariant should be chosen so that its expressions in terms of the different presentations directly relates to the aspects of the problem that one is interested to investigate. Normally, in order to extract a significant amount of information on a non-trivial mathematical problem, one single invariant will not suffice; one will have to consider several invariants (and hence generate several 'bridges') and combine the insights obtained thereby in order to eventually arrive at a global and deep understanding of the problem. In fact, each invariant will allow to 'read' (or 'decode') certain information 'hidden' (or 'coded') in the equivalence of toposes; there is not in general a privileged invariant that would subsume all the others, in the sense that the result generated by it through the 'bridge' technique (in the context of the given equivalence) would entail the results generated by the other invariants.

Any 'bridge' results in a connection between 'concrete' properties or notions involving the objects used for presenting the topos. Indeed, in spite of their crucial role in establishing such a correspondence, toposes do not appear in the final formulation of the result. It is important to bear in mind that the majority of correspondences, dualities and equivalences existing in mathematics are actually 'hidden' (in the sense that they are not induced by 'dictionaries' such as functors between sites or interpretations between theories and hence cannot be fully appreciated 'concretely') and manifest themselves only at the topos-theoretic level; our topos-theoretic reinterpretation and generalization of Galois theory (Caramello 2016c), reviewed in Caramello (2016a), is a compelling illustration of this (see also Sect. 9.12 below).

We should pause to note that this methodology represents a distinctly abstract way of doing mathematics, in the sense that it is an implementation of the principle according to which in order to obtain specific, 'concrete' information about a given mathematical problem, one should abstract it in several different directions (where by abstracting we mean focusing on a limited number of aspects at a time, temporarily forgetting about all the other ones) and then proceed to combine and integrate the insights obtained by investigating the resulting generalisations (or collecting information about them) to derive 'specific', 'concrete' results on the original problem. The underlying idea is that concreteness can be obtained in a top-down way by intersecting abstract planes, much as we can obtain a point by intersecting two lines in a plane or three planes in the three-dimensional affine space. The advantages of such an abstract approach are multiple. First of all, such a process creates a whole web of relations surronding the given problem, generating a sort of 'rain of results' falling into that territory. Moreover, it enlightens the general architecture of the proof and where and how the hypotheses come into play. This leads to a form of *modularity*: since the role of the hypotheses is enlightened, one is in the position to understand how different hypotheses could lead to different results, thereby also realizing a form of *continuity*. In other words, by setting the given problem within a family of related problems, this methodology allows to see it not in an isolated way but as part of a bigger picture, which greatly enhances one's understanding. Still, the 'bridge' technique is radically different from the traditional, relational way of doing mathematics based on category theory. The difference between our approach and the classical categorical one is particularly apparent in the treatment of duality theory (see Caramello (2020) for a discussion on this). Indeed, the fundamental relational notion in category theory is that of functor, i.e. morphism of categories; now, a functor is for us a kind of dictionary (it maps an object of the first category to an object of the second, and similarly for arrows) and, as we remarked above, functors are by no means sufficient to account for the majority of correspondences and dualities existing in mathematics. One needs a more flexible notion, and this is precisely what is achieved by the concept of a 'bridge' between different presentations of a topos.

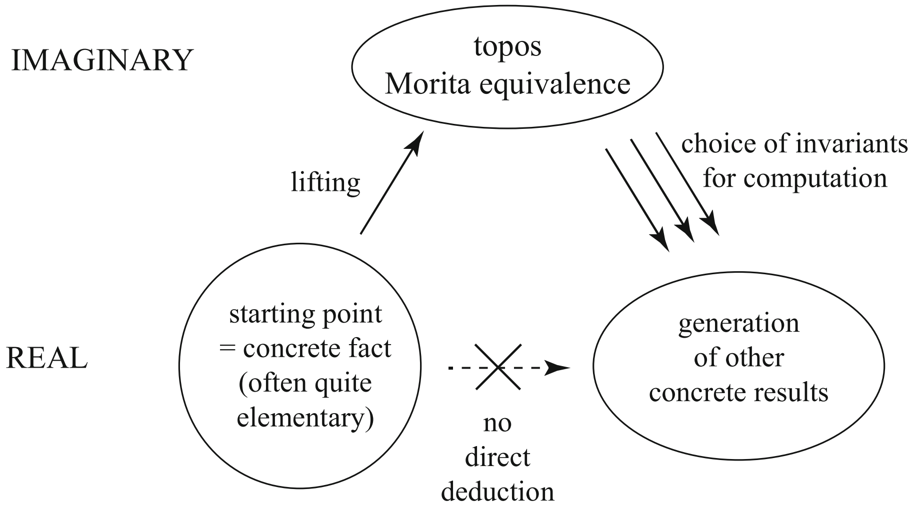
The methodology of toposes as 'bridges' represents a structuralist way of doing mathematics in the sense that it is based on uncovering hidden structures (that is, structures that are for the most part invisible from the concrete perspective of the 'working mathematician', such as Morita equivalences and topos-theoretic invariants) and letting them guide the mathematical exploration; it is therefore 'upside-down' with respect to the more classical mathematical styles based on a preliminary study of 'concrete' structures and to the subsequent, ad hoc identification of suitable invariants.

As far as the level of generality of the technique of topos-theoretic 'bridges' is concerned, this methodology is applicable to all those situations whose aspects that one wants to investigate can be encoded by means of suitable toposes and invariants on them. This certainly includes first-order mathematics (as formalized in terms of geometric logic) and a great amount of higher-order mathematics as well. Indeed, the possibility of considering Grothendieck toposes not just over the topos of sets but over an arbitrary Grothendieck topos allows one to construct classifying toposes for all those higher-order theories which can be formalized as (finite sequences of) relative geometric theories. Moreover, many mathematical objects of different kinds, including higher-order ones, can be used for presenting toposes; for example, the very notion of site is second-order.

9.12 The Duality Between 'Real' and 'Imaginary'

As already remarked in Sect. 9.10, the passage from a site (or a theory) to the associated topos can be regarded as a sort of 'completion' by the addition of 'imaginaries' (in the model-theoretic sense), which *materializes* the potential contained in the site (or theory). The duality between the (relatively) unstructured world of presentations of theories and the maximally structured world of toposes is of great relevance as, on the one hand, the 'simplicity' and concreteness of theories or sites makes it easy to manipulate them, while, on the other hand, computations are much easier in the 'imaginary' world of toposes thanks to their very rich internal structure and the fact that invariants live at this level.

The 'bridge' technique thus involves an ascent followed by a descent between two levels, the 'real' one of 'concrete' mathematics (represented by sites or other objects presenting toposes) and the 'imaginary' one of toposes, which we can schematically represent as follows:



This 'jump' from the 'real' into the 'imaginary is indispensable, or at least highly useful, in many situations to reveal correspondences between 'concrete' mathematical contexts that would be hardly visible otherwise (cf. Sect. 9.11). Toposes thus act as sorts of 'universal translators', crucial for establishing the 'bridges' but disappearing in the final formulation of results.

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