The “Unifying Notion” of Topos

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1 Introduction

I will talk about a fundamental concept introduced by Grothendieck, the notion of topos, and its unifying role in mathematics. In the title of this paper, the expression "unifying notion" is due to Grothendieck himself and can be found in *Récoltes et Semailles* [8], the famous autobiographical book of the mathematician in which he carries out a vast reflection on his mathematical work as well as on the reception of this work by the mathematical community of his time.

The theme of unification occupies an important place in this text and appears, in particular, in relation to the notion of topos as in the following remarkable passage:

> It is the topos theme which is this 'bed' or this 'deep river' where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the continuous and that of 'discontinuous' or 'discrete' structures.
>
> It is what I have conceived of most broad to capture with finesse, through a single language rich in geometric resonances, an 'essence' common to situations most distant from each other coming from one region or another of the vast universe of mathematical things.

The most striking feature of this extract is the universal dimension that Grothendieck attributes to toposes, a universality that is extremely rare at such a level in mathematics. It is indeed very difficult to be able to construct the same kind of object out of the most diverse mathematical situations, in such a way as to extract the essence of each situation. According to Grothendieck, this is what the toposes allow and realise.

As he announces, and we will come back to this, examples abound: toposes appear naturally in analytical, geometric, topological, algebraic, etc. frameworks, and, obviously, in connection with the theory of categories, toposes being themselves categories. This is actually one of the most profound unifications established by the notion: that between category theory and topology through Grothendieck's primary and fundamental intuition interpreting any topological space through the intermediary of its category of sheaves of sets. Historically, it was this intuition that led Grothendieck to introduce toposes and since then it has emerged from various landscapes, through various constructions, to an impressive level of generality since, for example, we can associate toposes with mathematical theories, in the sense of logic.

It is therefore natural to ask the following question: *Since toposes can be associated in an essential way with such a variety of situations, can this unification take on a substantial and dynamical meaning?*

More specifically, would it be possible to use this unification to achieve **transfers** of knowledge from one context to another?

This question has been—and still is—my main concern since my doctoral research until today: trying to realise this aspiration of unification through methods that would allow to make this unification a technical tool establishing 'bridges' between different theories and thus allowing new approaches to given mathematical problems.

Before going into detail, I would like to introduce the themes we will be discussing. First of all, I would like to talk about the multiform nature of toposes. Indeed, toposes turn out to be central objects in mathematics, in the sense that they can be profitably considered from several different viewpoints. There are three classical points of view:

– That of a topos as a metamorphosis of the concept of space, that is, Grothendieck’s original conception
– The one introduced by logicians who interpret toposes as mathematical universes with an internal logic of their own, in which it is possible to consider models of mathematical theories such as first-order theories or even higher-order theories and to try to classify these models
– Lastly, that of toposes seen as classifying spaces for certain types of structures, a theme introduced informally by Grothendieck and clarified thanks to the work of categorical logicians in the 1970s

I will then briefly discuss the reception of the notion of topos by the mathematical community and, in particular, the reception of the unifying vision that Grothendieck foresees. This theme will prove interesting from both a philosophical and sociological point of view. In this regard, I have selected a few quotations from *Récoltes et Semailles* in which Grothendieck talks about the difficulty that the mathematical community has had—and in part continues to have—in hearing the unifying message he brought. In particular, we will ponder on which were the obstacles to the understanding of toposes as unification tools as opposed to their understanding as technical tools, for example, in relation to questions—also motivating for Grothendieck—related to étale cohomology. It will be seen that, to Grothendieck’s visionary aspect, the mathematicians of his time preferred the ‘pragmatic’ dimension of toposes having as its sole aim that of solving difficult problems.

I will then give an idea of the use that can be made of Grothendieck toposes as ‘unifying bridges’, i.e. as objects which can be used for making transfers between different mathematical theories or situations, as I have conceived and developed it since my doctoral studies. I will give both the key principles of this approach, which seems to me to extend Grothendieck’s approach in a natural way, and some concrete examples.

2 The Multiform Nature of Toposes

Since their introduction by Grothendieck in the 1960s, toposes have appeared, with various approaches, in the categorical interpretation of various theories at the borders of geometry, algebra and logic. As we have already mentioned, a Grothendieck topos can alternatively be interpreted as a generalised topological space, as a mathematical universe or as a theory modulo a certain equivalence relation called 'Morita equivalence'.[1] Let us briefly recall each of these different points of view.

2.1 Toposes as Generalised Topological Spaces

On this subject, we must begin with a clarification. Grothendieck, rather than talking of generalised spaces, speaks of a 'metamorphosis' of the notion of topological space. Indeed, toposes are not technically a strict generalisation of topological spaces; for a true equivalence, we must restrict ourselves to a particular class of topological spaces, called *sober* spaces. Fortunately, otherwise their interest would have been attenuated, sober spaces form a large class of topological spaces which includes most of those which appear naturally in mathematics. Toposes generalise in a strict sense sober topological spaces.[2]

In any case, it is not really the dimension of generalisation that is interesting but rather that of metamorphosis: the fact of being able to think of topological spaces in category theory. This has enormous consequences since, from a technical point of view, a topological space is something very different from a category. Especially since, in terms of structure, toposes are extremely rich categories, whereas a 'mere' topological space leaves little room for algebraic or structural manipulations. With toposes, we really change from one world to another and the term 'metamorphosis' chosen by Grothendieck is particularly appropriate.

This first approach to the notion of topos was introduced by Grothendieck in the early 1960s in the Bois Marie seminar and appears in the first volume of SGA 4. In this book, we find the development of the foundations of the theory, which is based on several key ideas, the first of which, that of the sheaf, goes back to Leray. It is a question of thinking about topological spaces through their associated categories of sheaves. According to Grothendieck, the notion of topos appeared by going 'all the way to the end' ('*jusqu'au bout*') of Leray's intuition. The leading idea is the following.

Given a topological space X, Grothendieck associates with it its category $\mathbf{Sh}(X)$ of sheaves. This is a category whose objects are the sheaves of sets on X and whose morphisms are the natural transformations between its sheaves seen as functors. The depth of this idea is easy to illustrate: first of all, the fundamental topological properties of X–connectedness, compactness, etc.—can be 'read' from $\mathbf{Sh}(X)$[3],

[1] This is the relationship that identifies two theories precisely when they have the same classifying topos.

[2] X is sober if for any non-empty closed set A that cannot be written as a proper union of two closed sets, there exists a unique point x such that A is the smallest closed set containing x.

The topos of sheaves on a sober topological space determines this space, to the nearest homeomorphism. It is in this sense that we can say that the notion of topos generalises that of topological space.

[3] '**Sh**' for *sheaf*.

so no topological information is lost when passing from a topological space to its category of sheaves of sets. Secondly, we gain a lot from the point of view of structures because the sheaf categories associated with topological spaces have remarkable structural properties: they have all limits and colimits, function spaces (also called 'exponentials') of an object by another and also an object classifying the subobjects in the category.

From a structural and algebraic point of view, one could speak of $\mathbf{Sh}(X)$ as an 'ideal world': everything exists, one can form quotients, spaces of functions, coproducts, etc., in other words everything one is used to doing in the classical context and everything that we are used to doing in the classical context of sets (provided, however, that we do not use non-constructive principles, but we will come back to this). In short, starting from a relatively 'unstructured' topological space, we end up with a 'hyper-structured' and even 'maximally structured' object, an illustration of the 'metamorphosis' of which Grothendieck speaks!

But he was not satisfied with considering categories of this form; he also thought deeply about the notion of sheaf itself. He wondered whether this notion only made sense in the context of topological spaces or, potentially, in another, broader one, which went well beyond topology. Of course, the primary motivation for this questioning came from Weil's conjectures, from the need to define non-classical cohomological theories, going beyond the topological framework. Thus, Grothendieck, knowing the easiness of defining cohomology from the category of sheaves of sets of a topological space, realised that a more general notion of sheaf would make it possible to define new cohomological theories, with the firm hope of finding, among them, those that would help prove Weil's conjectures. And this is precisely what happened.

Before going further into Grothendieck's reflections, let us take the time to remind ourselves of the notions of presheaf and sheaf. If X denotes a topological space, $\mathscr{O}(X)$ the category of opens of X and **Set** the category of sets, then a presheaf on X is a contravariant functor

$$\mathscr{O}(X) \longrightarrow \mathbf{Set}$$

or, equivalently, a covariant functor of the opposite category

$$\mathscr{O}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}\ .$$

Thus, in order to define presheaves on X, we only need the category $\mathscr{O}(X)$ whose objects are the opens of the space X and whose morphisms are the inclusions between open sets. This definition turns out to be particularly malleable and can be generalised without any difficulty to categories. If $\mathscr{C}$ denotes any category, a presheaf on $\mathscr{C}$ is a covariant functor:

$$\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}\ .$$

The notion of sheaf of sets on a topological space X, on the other hand, is based on the notions of covering family and glueing. A covering of an open set U in the usual topological sense is a family of open subsets $(U_i)_{i \in I}$ such that each U_i is contained in U and that $\cup_{i \in I} U_i = U$. A sheaf on X is then a presheaf such that giving a section on an open U covered by open subsets U_i is equivalent to giving a family of sections on the U_i which match on the intersections $U_i \cap U_j$.

The generalisation of the notion of sheaf to categories requires a categorical translation of the notion of covering family and of the glueing condition. It was formulated by Grothendieck who introduced the notion of Grothendieck topology on a category. If $\mathscr{C}$ denotes a small category, i.e. a category whose objects and arrows form a set, and if c is an object of $\mathscr{C}$, a covering family of c will be nothing but a family of morphisms $c_i \to c$ possessing properties analogous to those of covering families of open sets of a topological space. More precisely, one requires that any object c of $\mathscr{C}$ be associated with a collection $J(c)$ of elements called *sieves* on c, which are sets S of morphisms of $\mathscr{C}$ having c as their target and closed under composition on the right, i.e. such that whatever f is in S and g is a morphism of $\mathscr{C}$ composable with f, $f \circ g \in S$.

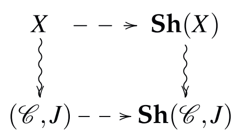
A *Grothendieck topology* on $\mathscr{C}$ is then a function J assigning to any object c of $\mathscr{C}$ a set $J(c)$ of sieves on c in such a way that the following properties are satisfied:

- (Maximality axiom) For any object c of $\mathscr{C}$, the maximal sieve on c consisting of all the arrows with target c belongs to $J(c)$.
- (Pullback stability Axiom) For any arrow $f : d \to c$ in $\mathscr{C}$ and any sieve S in $J(c)$, the sieve $f^*(S) = \{g : e \to d \mid f \circ g \in S\}$ belongs to $J(d)$. In the topological context, this is indeed what happens by distributivity of arbitrary unions with respect to finite intersections.
- (Transitivity) For any sieve S on c and any $T \in J(c)$, if $f^*(S) \in J(\mathrm{dom}(f))$, for any $f \in T$, then $S \in J(c)$.

These conditions are both simple and natural, and this explains why one easily encounters, just about everywhere in mathematics, categories that can be provided with Grothendieck topologies. Such a topology is what is needed to define sheaves on the relevant category:

Definition A sheaf on a pair $(\mathscr{C}, J)$ formed by a small category $\mathscr{C}$ and by a Grothendieck topology J on $\mathscr{C}$ is a presheaf $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ such that, for any sieve $S \in J(c)$ and any family $\{x_f \in P(\mathrm{dom}(f)) \mid f \in S\}$ satisfying $P(g)(x_f) = x_{f \circ g}$ for any $f \in S$ and any arrow g of $\mathscr{C}$ composable with f, there exists a unique element $x \in P(c)$ such that $x_f = P(f)(x)$ for every $f \in S$.

The category of sheaves thus defined will be denoted by $\mathbf{Sh}(\mathscr{C}, J)$.



Finally: **a Grothendieck topos is a category equivalent to a category of sheaves of the form Sh**$(\mathscr{C}, J)$.[4]

We note that a pair $(\mathscr{C}, J)$ as in the above definition is called a *site* by Grothendieck. This terminology, as always with him, is subtly chosen since it expresses the *contingent* character of this notion in relation to the *invariant* dimension of toposes. In a way, a site would only be a sketch, a presentation of a situation, whereas the topos is the object that extracts the abstract essence of the situation.

And, indeed, a site is not an object in itself, it is a pair, like an artist's partial sketch, which is coherently completed by the construction of the associated topos from the site's data.

We shall see later, and this is what will allow us to use toposes as 'bridges', that different sites can generate equivalent toposes (in the sense of category theory). In this respect, Grothendieck makes an analogy, admittedly reductive but formally correct, with groups[5] which, as is well known, can be presented in different ways by generators and relations. The same applies to toposes, which have several presentations—in fact, an infinite number—by different sites, still with much more expressiveness and freedom for sites, which are found everywhere in mathematics and logic in extremely varied forms.

The fact that different sites can generate equivalent toposes is very interesting when we realise that, since obtaining invariants by categorical equivalence is almost trivial (any property or notion naturally formulated in categorical language necessarily being invariant), we have, in principle, an enormous quantity of invariants that can be defined on toposes, well beyond the classical invariants such as cohomology. Any invariant can thus be considered and studied in terms of different presentations by sites, which gives rise to a veritable mathematical morphogenesis. Indeed, when we study the different ways of expressing the same invariant on various toposes through different presentations, we uncover a truly astounding mathematical wealth. We shall give some examples below.

Before considering the logicians' approach to toposes, it is interesting to see how Grothendieck expresses himself about the birth of the idea of topos. In his literary texts, *Récoltes et Semailles* or *La clef des songes*, he attaches great importance to childhood in his vision of life, mathematics and creation. In the following excerpt, he returns to the 'childlike', innocent idea of topos, that innocence which is the key to creativity, though unequally shared by the colleagues of his time:

> Like the very idea of sheaf (due to Leray), or that of scheme, like any 'great idea' that comes to upset an inveterate vision of things, the idea of topos is disconcerting because of its character of naturalness, of 'self-evidence', by its simplicity (almost, one would say, verging on the naive or the simplistic, or even the 'dumb') – by that special quality which so often makes us exclaim: 'Oh, that's all there is to it!', with a half-disappointed, half-

[4] Intrinsic characterisations of toposes exist—as shown by Giraud, for example—but we will not use these results here.

[5] Although reductive, the analogy is not without foundation. Indeed, Grothendieck himself envisaged and studied toposes as a generalisation of groups.

> envious tone; with, perhaps, also an undertone of 'wacky', of 'not serious', which one often reserves to anything that confuses by an excess of unexpected simplicity. To what reminds us, perhaps, of the long-buried and disowned days of our childhood...

Grothendieck was aware of the elegance, simplicity and naivety of the notion he was introducing, but, at the same time, he recognised its disconcerting character, because of the unexpectedness it represented in the eyes of the mathematical community of his time.

In fact, Grothendieck discusses at length the psychology behind the rather virulent and often irrational hostility—which persists in some circles even today—towards the notion of topos; we will return to this later. He notably adds:

> Nevertheless, I can't think of anyone else on the mathematical scene, over the last three decades, who could have had this naivety, or this innocence, to take (in my place) this other crucial step of all, introducing the so childish idea of topos (or at least that of 'site').

2.2 Toposes as Universes

This approach was proposed a few years after Grothendieck's work by W. Lawvere and M. Tierney. Their school studied toposes axiomatically, as categories within which one can 'do mathematics', thanks in particular to categorical semantics. They thus introduced the novel point of view on toposes as *alternative* mathematical universes, with their own internal logic, in which mathematics can be developed.

In order to give an impressionistic, non-technical picture, inspired by the logical point of view on mathematical universes, we will say that toposes resemble, *in certain aspects*, the classical universe in which we are used to working and which is the universe of sets. Thus, as far as pure computations, algebra, are concerned, toposes behave similarly to the universe of sets: one finds in both fundamental formal properties such as the existence of limits and colimits.

Nevertheless, there are essential differences which distinguish them, first of all their internal logic: for example, the notion of truth in a topos is not generally Boolean[6] and one can encounter a great variety of truth values and therefore much richer logical phenomena. Logics can be very different from one topos to another. They have the fertility of intuitionistic, constructive logics, as opposed to the relative 'poverty' of Boolean logic.

Furthermore, it is possible, thanks to the rich categorical structure that each topos possesses, to consider models of mathematical theories, for example, first-order ones, within any topos. This has immense consequences, such as allowing the development of a *functorial model theory*, on which we shall speak later. It is certain, in any case, that the potential field of study of logicians specialised in model theory and who, until then, only worked in the topos of sets has been considerably

[6] See the presentation by Alain Connes in the volume *Lectures Grothendieckiennes* (Spartacus IDH and SMF, 2022).

expanded. Today, one can consider theories in any topos, study their properties in these new frameworks, vary the ambient toposes, link them functorially and thus open up new horizons.

This great variety, this great flexibility, offers the possibility of constructing new mathematical worlds which allow to accommodate (in the sense of giving substance to) concepts which would not exist or could not perhaps be embodied in a constructive or natural way in the classical foundations of mathematics. I am thinking, for example, of infinitesimals or of problems of independence of axioms in set theory, or even of 'forcing'. These questions can be tackled very satisfactorily in the framework of toposes, precisely by exploiting the flexibility that these offer to generate new landscapes.

Once a toposic world has been constructed in which the problem we are interested in has a realisation, an expression, we can always envisage the possibility of constructing, from this topos, other toposes (or objects) that respond to this problem and provide a framework closer to our intuition. For all that, the existence—proven—of such a 'solution' topos is remarkable in itself, since it offers us a universe in which, to take up one of the cited examples, the question of the existence or not of infinitesimals does not arise: they exist and not only in an isolated topos. Perhaps we will need the existence of these infinitesimals in a context verifying certain properties, but then the problem becomes more relational than absolute: it becomes that of transferring to our particular context the properties of a topos in which these infinitesimals exist. We can thus see that the theory of toposes realises an enlargement of the mathematical ontology which is not only conceptual, but technically extremely powerful.

The possibility of functoralising model theory through topos theory is, in fact, a striking illustration of this technical power. In set theory there is a classical notion of model for first-order theories, due to Tarski: sorts, i.e. names of different types of objects, are interpreted by sets, function symbols by functions and relation symbols by subsets. This yields a structure. A model is then nothing else than a structure in which all the axioms of the theory are satisfied.

This notion can be generalised to any topos; the sorts will be interpreted by objects of the topos, the function symbols by arrows and, finally, the relation symbols by subobjects. As for the structural dynamics generated, it is perfectly similar to the classical case (see [9, 11] or [6]). Thanks to the categorical richness of toposes, one can then inductively define the interpretation of the formulae and, with it, the toposic models of any theory, not only of first order, although we confine ourselves here to this case. This makes it possible to '*relativise*' mathematics, in the sense of developing it in relation to variable bases: we do not have a single universe at our disposal in which we would be obliged to view the problem, we can change the universe. This adds a new dimension to mathematics.

Let us now return to a point raised earlier, which deserves some clarification. Faced with the immense variety of toposes, the natural questions to ask in a particular context are: 'Is the topos of sets the most intuitive, the most appropriate, the most natural for considering this context? Is there not a topos that offers a privileged point of view that would shed more light on the problem under consideration?' We answer this last question in the affirmative, and, indeed, the theory of classifying topos—the subject of the following paragraph—allows us to give a precise mathematical meaning to this intuition: there is always a privileged point of view that we can have on a certain theory.[7] Thus, with the help of functorial model theory, we can classify models. Now, in studying the relationship between validity in models and demonstrability in the theory, if we restrict ourselves to set-based models, we come up against the following problem: Gödel's completeness theorem affirms that if a first-order theory is finitary *and if we decide to accept the axiom of choice*, then the validity of a closed first-order formula in all the set-based models is equivalent to demonstrability in the theory. But, indeed, in order to prove this, we need to force, i.e. to use a non-constructive principle, namely, the axiom of choice. This gives an indication of the 'deficiencies' of set-based models, in which syntax and semantics do not merge.

If, on the other hand, we decide to widen the field of study to the context of toposes, we find a particular topos, with a particular model, the *universal model* of the theory, in which a unification between syntax and semantics is realised. Indeed, what is valid in this model can be demonstrated in the theory and, of course, vice versa (and all of this in an entirely constructive way). So, rather than considering all the set-based models of the theory in an attempt to obtain a faithful representation of it, one can concentrate on this perfectly constructive universal model.

This is an illustration of the fact that broadening the mathematical framework brings great rewards. Once the right point of view has been identified, that of the universal model living in the classifying topos of the theory, one can allow oneself to revisit all the set-based models as points of the classifying topos and thus as 'deformations' of this universal model.

2.3 Toposes as Classifying Spaces

The intuition for this last point of view is, once again, due to Grothendieck, although he did not push it all the way through, as we shall see, particularly in its technical realisation. According to this perspective, toposes are regarded as objects that embody the mathematical or semantic content of first-order theories of a particular form: those that can be formulated within 'geometric logic". Every such theory has an associated topos, called its classifying topos, which, as the name suggests, classifies its models in any topos. The equivalence relation that identifies two theories when they have the same classifying topos is called *Morita equivalence*. Any topos is the classifying topos of a theory (and, in fact, of an infinite number of theories). We can then think of a Grothendieck topos as an equivalence class of geometric theories modulo this Morita equivalence relation.

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[7] At least for first-order ('geometric') theories

What Is the Idea Behind the Notion of Classifying Topos? Grothendieck has always attached great importance to the representability paradigm and has given it a central role in his work in algebraic geometry. In this sense, he builds on the idea of Yoneda and his famous categorical lemma: to think of an object, in a category, through the morphisms of arbitrary objects in the category towards that object. The philosophy of the Yoneda lemma thus leads us to think of an object as the collection of its generalised elements. In line with this paradigm, Grothendieck was interested in toposes from the point of view of the structures that they classify, and, to this end, he had to consider a *category of toposes*. The objects of this category are, of course, toposes. As for morphisms, Grothendieck defines them as follows: a *(geometric) morphism* of toposes is a pair $f := (f^*, f_*)$ of adjoint functors, more precisely a direct image functor f_* (right adjoint) and an inverse image functor f^* (left adjoint), with the condition that the inverse image functor *preserves finite projective limits*.[8]

In the spirit of classifying spaces, the idea is then to understand, in this category, a given topos $\mathscr{E}$ by considering the geometric morphisms of any topos $\mathscr{F}$ into $\mathscr{E}$, with the aim of also determining the topos $\mathscr{E}$ up to equivalence. This idea of classification makes it possible to develop a new point of view on toposes, which we will explain in more detail below. But first, let us return to Grothendieck. In various places in his work we find considerations about toposes as classifying spaces. However, the most explicit text on this subject is the thesis of his student M. Hakim ‘Topos annelés et schémas relatifs’ in which this view of toposes as classifying spaces is implemented in particular cases. Four toposes are studied in this volume as classifying toposes of certain types of rings; for example, it is shown that the Zariski topos classifies local rings, the étale topos classifies strictly Henselian rings, etc.

We then find more general considerations by Grothendieck on this subject, particularly in SGA 4—in particular, sentences where he refers to the classifying topos of structures ‘expressed in terms of finite projective limits and inductive limits of any kind’. Of course, this is a rather vague expression—he realises this—and the problem of formalising this intuition arises.

If the intuition is entirely correct, it was only materialised with the help of the language of logic, which, most probably, he did not know. Indeed, according to a classical paradigm of logic, formulae are constructed inductively from simple formulae, which must be compared to Grothendieck’s explicit references to operations that are repeated (finite projective limits and arbitrary inductive limits). He writes in SGA 4 (we highlight in bold some parts):

> The exactness properties of the inverse image functor u^* of a geometric morphism of toposes $u : \mathscr{E} \to \mathscr{E}'$ ensure that for any kind of algebraic structure[9] L whose data can be described in terms of data of arrows between basic sets and sets deduced from these by

[8] Note that, since the inverse image is a left adjoint functor, it automatically preserves arbitrary inductive limits, whereas in general it does not preserve any projective, or even finite, limits. This is therefore a condition that one imposes.

[9] Here, Grothendieck has in mind algebraic structures of which he gives a few examples at the end of the extract, and not algebraic algebraic structures in the sense of the universal algebra of logic.

> repeated applications of finite projective limit operations and any inductive limits, and for any 'object of $\mathscr{E}'$ with a structure of species L', its image by u^* is endowed with the same structure. Rather than entering into **the uninvolving task of giving a precise meaning to this statement and justifying it formally**, we advise the reader to make it explicit and to convince themselves of its validity for species of structures such as group, ring, module over a ring, comodule over a big ring, etc. comodule over a ring, bigebra over a ring, torsor under a group.

In terms of mathematical content, this means that for two toposes $\mathscr{E}$ and $\mathscr{F}$ and a geometric morphism $u : \mathscr{F} \to \mathscr{E}$, the inverse image functor u^* transforms any L-structure of $\mathscr{E}$ into an L-structure of $\mathscr{F}$.

This is an absolutely fundamental remark which, once formalised, makes it possible to define a '(pseudo)functor of the models of a theory', to use the logical terminology. More precisely, we find the functoriality of models mentioned in the previous paragraph. In particular, this means that instead of having a single context in which to consider the models, we can vary our topos and, in doing so, define a (pseudo)functor. The question of classifying topos then becomes that of the representability of this (pseudo)functor. If it is representable, we say that the object representing it is the *classifying topos* for that theory.

Moreover, a representable functor gives not only an object of the category under study but also an element of the value of this functor in this object: in the framework we are interested in, this is what we will call the *universal model of the theory*.

Reading this extract, it is clear that for Grothendieck the intuition was very clear, but he still felt the need for a better formalisation, and it is precisely this 'not very engaging' (but very interesting) task of formalising 'geometric logic' that the categorical logicians in the 1970s, in particular W. Lawvere, M. Makkai, G. Reyes, A. Joyal, J. Bénabou and J. Cole, accomplished.[10]

More precisely, in the language of logic, any (first-order) geometric theory $\mathbb{T}$ can be canonically associated with a topos $\mathscr{E}_{\mathbb{T}}$, called its *classifying topos*—in the sense that it represents its (pseudo)functor of models—and which embodies its 'semantic core'.

The topos $\mathscr{E}_{\mathbb{T}}$ is characterised by the following universal property: for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$, where $\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}})$ denotes the category of geometric morphisms $\mathscr{E} \to \mathscr{E}_{\mathbb{T}}$ and $\mathbb{T}$-mod$(\mathscr{E})$ is the category of models of $\mathbb{T}$ in $\mathscr{E}$.

So a topos that verifies this universal property is uniquely determined up to categorical equivalence. We can therefore speak of *the* classifying topos of a geometric theory.

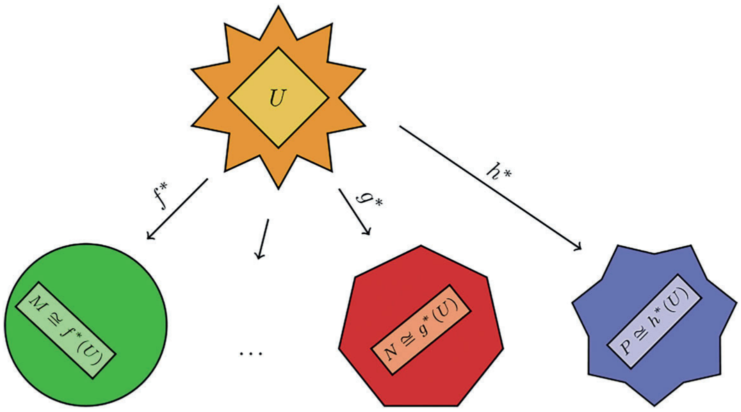
In the figure below, the coloured geometric shapes represent toposes in which models U, M, N and P of the theory live; in particular, the yellow diamond

[10] See, for example, [11].

represents the classifying topos of the theory and U the universal model inside it. This model is the image of the identity of $\mathscr{E}_{\mathbb{T}}$ under the equivalence

$$\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E}) \ .$$

The models M, N, P, etc. are thus all images of U by inverse image functors f^*, g^*, h^*, etc. of geometric morphisms f, g, h, etc. from the toposes where the models live to the classifying topos $\mathscr{E}_{\mathbb{T}}$. They therefore appear as ‘deformations’ of the universal model of the theory.



The figure highlights the fact that the classifying topos of a theory is the privileged place where the symmetries of the theory unfold, its ‘centre of symmetry’. It constitutes the space where the fundamental invariants of the theory (those which depend solely on its semantic content, independently of its different syntactic presentations) are naturally defined. We can also see the advantage of having extended our context for considering models of the theory from the usual set-theoretic framework to that of toposes. Indeed, such a classification result for models does not exist in the restricted context of sets.

We shall not enter into the technical details of the definition of geometric logic; let us just say that a geometric theory is a theory written in a first-order language whose axioms are all of the form

$$(\phi \vdash_{\vec{x}} \psi),$$

where ϕ and ψ are *geometric* formulae, i.e. obtained from atomic formulae using only finitary conjunctions, infinitary disjunctions (indexed by an arbitrary set) and existential quantifications. The expression $(\phi \vdash_{\vec{x}} \psi)$ means that for all values of the variables in $\vec{x}$, $\phi(\vec{x})$ implies $\psi(\vec{x})$. Note that any inverse image functor f^* of a geometric morphism sends models of a geometric theory $\mathbb{T}$ into models of $\mathbb{T}$, since the interpretation of atomic formulae and finitary conjunctions is expressed in terms of finite limits, whereas infinitary disjunctions and existential quantifications are interpreted in terms of arbitrary colimits and finite limits.

The apparently very particular and restrictive form of the axioms of a geometric theory should not mislead as to the level of generality of this logical system. In fact, thanks to a process called *Morleyisation*, we can canonically associate with any first-order finitary theory a finitary geometric theory with essentially the same set-based models. Moreover, the infinitary nature of geometric logic makes it possible to formally express and study many natural properties that do not admit of finitary axiomatisations (think, for instance, of the property of an element of a ring to be nilpotent, which can be expressed in an infinitary but not finitary way in the language of ring theory, or the notion of a strong unit for a lattice-ordered group, etc.).

As one can easily imagine, geometric theories arise in all areas of mathematics. The existence theorem for classifying toposes is therefore an extraordinarily general result, since it states that all (pseudo)functors of the form $\mathscr{E} \rightsquigarrow \mathbb{T}\text{-mod}(\mathscr{E})$, where $\mathbb{T}$ is any geometric theory, are representable. Usually in mathematics, showing that certain functors which are rich in information are representable is exceptional and has far-reaching consequences. Think, for example, of scheme theory, where representability theorems are relatively rare and important. Here, on the other hand, the existence of a classifying topos that represents the functor of models of a geometric theory is known in advance, so that refining and deepening this knowledge consists in studying this classifying topos and its properties. We are no longer dealing with a problem of existence, but with the study of an object whose existence has been established. On the other hand, the notion of (geometric) morphism of toposes is particularly flexible, which means that there are ‘enough morphisms’ to represent rich functors such as those of the models of a geometric theory; note that, on the other hand, the notion of morphism of schemes is more rigid.

3 The Reception of Toposes

Grothendieck complains repeatedly in *Récoltes et Semailles* about the bad reception of toposes in the mathematical community, which he attributes mainly to the lack of vision of his former colleagues and students. He writes, for instance:

> I gradually came to realise, though I’m not sure how, that several notions that were part of the forgotten vision had not only fallen into disuse, but had become, in a certain beau monde, the object of condescending disdain. Such was the case, in particular, of the crucial unifying notion of topos, at the very heart of the new geometry — the very notion that provides the common geometric intuition for topology, algebraic geometry and arithmetic — the very notion that enabled me to develop both the étale and the ℓ-adic cohomological tool, and the key ideas (more or less forgotten since, it is true) of crystalline cohomology. To tell the truth, it was my very name, over the years, which insidiously, mysteriously, had become an object of derision — as a synonym for muddy bombast ad infinitum (such as that on the famous ‘toposes’, indeed, or these ‘motives’ that he was raving about and that nobody had ever seen. . .), splitting hairs over a thousand pages, and gigantic chatter about

> what, in any case, everybody had already known all along and without having waited for it....

Such an irrational reaction to a profound and fruitful concept like that of topos might seem implausible. And yet Grothendieck’s analysis is sadly lucid. I can also say this in the light of my own personal experience as a researcher working on toposes with the aim of exploiting their unifying power (30 years after these statements!)—for years I have been the victim of exactly the same kind of denigrations as those denounced by Grothendieck, to the point that I was obliged to undertake a public initiative of clarification to show the ill-foundedness of such accusations and restore my scientific reputation:[11] as it happens, the term which was used to discredit some of my work as something that, ‘in any case, everyone had already known all along and without having waited for it’ was the highly ambiguous and dangerous term of folklore.[12]

Here are some more excerpts from *Récoltes et Semailles* about the thwarted reception of toposes:

> For fifteen years (since I left the mathematical scene), the fertile unifying idea and the powerful discovery tool that is the notion of topos, has been maintained by a certain vogue in the banner of reputedly serious notions. Even today, very few topologists have the slightest suspicion of this considerable potential expansion of their science, and of the new resources it offers.

This extract is particularly interesting because it shows Grothendieck’s conception of toposes as objects not only of mathematical nature but also of *meta-mathematical* one, as notions capable of guiding the mathematical exploration and leading to the introduction of new concepts and results.

In the following passage, Grothendieck highlights the abstract nature of the concept of topos as an explanation for the reluctance of mathematicians to take it seriously:

> Given the disdain with which some of my former students treated this crucial unifying notion, the latter has seen itself condemned to a marginal existence since I left. [...] toposes are nevertheless encountered at every step in geometry — but we can of course do without seeing them, just as we have done for millennia without seeing groups of symmetries, sets, or the number zero.

This extract appears all the more remarkable when one considers that, from a logical point of view, the passage from a site to the corresponding topos can be described as a completion through the addition of ‘imaginary’ objects (in the sense of model theory), similar to what happens when we complete a numerical system, such as the set $\mathbb{N}$ of positive integers into the group $\mathbb{Z}$ of relative numbers, or the

[11] The interested reader can find more information on this controversy at https://www.oliviacaramello.com/Unification/InitiativeOfClarificationResults.html.

[12] Readers wishing to delve deeper into the sociological and ethical dimension of this type of accusation are referred to the article ‘Epistemic injustice in mathematics’ by C.J. Rittberg, F. Stanley Tanswell and J-P. Van Bendegem (*Synthese*, Springer, 2018), which analyses in particular my personal case.

ring $\mathbb{Z}$ into the field $\mathbb{Q}$ of rational numbers, or the field $\mathbb{R}$ of the real numbers into the complex plane $\mathbb{C}$. In each of these cases one adds entities whose nature may seem more abstract than that of the initial ones, but the interest of such constructions lies mainly in the new computational possibilities that they open up, resulting from the existence of more *structures* and *symmetries* in the extended context than in the original one (think, e.g. of symmetry with respect to zero in the relative numbers or of the fundamental theorem of algebra, for which there is no analogue in the restricted context of real numbers, or the fact that $\mathbb{N}$ is merely a monoid, whereas $\mathbb{Z}$ is a ring and $\mathbb{Q}$ is a field). Grothendieck ironises that we can of course do without seeing these imaginary objects and pretend that they do not exist, with the result of unknowingly depriving ourselves of all the conceptual and technical resources that they offer. Indeed, if we look at the school of algebraic geometry founded by Grothendieck in the decades following the Master's departure, we see that there has been a very sophisticated use and development of the cohomological techniques forged by Grothendieck without this being accompanied by a systematic development of the underlying theory of toposes. In fact, most of the geometers 'heirs' of Grothendieck have privileged sites, as objects that concretely express the geometric content of a situation, and the cohomological invariants of the associated categories of sheaves while 'neglecting' toposes, which are nevertheless the generalised spaces on which these invariants are naturally defined. We will come back to this point later.

3.1 The Vision and the Tool

The following excerpts highlight this dichotomy between the reception of the cohomological tools forged by Grothendieck and the recusation of the vision that inspired their development:

> The set of two consecutive seminars SGA 4 and SGA 5 (which for me are like **a single** 'seminar') develop from nothing, both the powerful instrument of synthesis and discovery represented by the language of toposes, and the perfectly perfected and perfectly effective **tool** that is étale cohomology — better understood in its essential formal properties, from that moment on, than was even the cohomological theory of ordinary spaces.

> These two seminars are for me indissolubly linked. They represent, in their unity, both the **vision**, and the **tool** — toposes, and a complete formalism of étale cohomology. While the vision is still rejected today, the tool has, for more than twenty years, profoundly renewed algebraic geometry in its aspect which, for me, is the most fascinating of all — the 'arithmetic' aspect, apprehended by an intuition, and by a conceptual and technical baggage, of 'geometrical' nature.

> The operation 'Cohomologie étale' consisted in **discrediting the unifying vision** of toposes (such as 'nonsensefg, bombinage, etc.)... and on the other hand, to **appropriate the tool**, i.e. the authorship of the ideas, techniques and results that I had developed on the theme of étale cohomology.

3.2 'Sites Without Toposes', 'Toposes Without Sites'

As mentioned above, most algebraic geometers after Grothendieck have essentially abandoned the notion of topos by focusing on the study of particular cohomological theories associated with specific geometric sites, probably out of a concern for pragmatism. This practice of neglecting toposes in favour of sites—which can be summed up by the formula '*sites without toposes*'—has been largely shared within this community. As a consequence, even in the particular context of cohomological invariants, the new computational possibilities resulting from the existence of different presentations for the relevant toposes, pertaining in principle to different areas of mathematics (see the next section for more information on the technique of toposes as 'bridges'), have not been exploited as much as they could have been.

On the other hand, categorical logicians, after defining geometric logic in the 1970s, essentially abandoned the study of classifying toposes, to focus on other themes such as that of 'elementary toposes' of W. Lawvere and M. Tierney. This is a type of category that differs from the that of Grothendieck toposes notably by being finitely axiomatisable in the language of categories. However, elementary toposes do not have all colimits and they are not always representable by sites (by definition, a topos which admits a site of definition is a Grothendieck topos).

Such has been, in this school, the determination to abandon sites in favour of an exclusively axiomatic approach to toposes based on finitary logic, and the conviction that this theory should have replaced Grothendieck's theory of toposes, that, in the literature it has produced, the very term *topos*, due to Grothendieck and used until then to designate the concept of a category equivalent to a category of sheaves (of sets) on a site, has started[13] to designate a different and more general concept, that of cartesian closed category with finite limits and a subobject classifier (what we now call 'elementary topos' to distinguish it from Grothendieck's notion of topos). Note that in an arbitrary 'elementary topos' one cannot properly do neither geometry nor topology, since this type of category does not necessarily have arbitrary colimits. Nor is there any duality between this class of categories and a class of first-order theories, as in the case of classifying toposes. In fact, elementary toposes classify the models of higher-order intuitionistic type theories, albeit in a rather rigid manner as the functors used in this classification are those that preserve the entire 'logical' structure of these categories (exponentials and the subobject classifier); indeed, these functors, unlike geometric morphisms of toposes (which are induced by any morphism or comorphism of sites, and in particular by any continuous application of topological spaces and any group homomorphism), do not arise naturally in the mathematical practice.

The choice to study toposes without reference to their presentations (an approach that could be summarised by the formula '*toposes without sites*') has deprived this school of the possibility of obtaining deep *applications* of toposes in 'concrete'

[13] In an inappropriate manner, both from a historical and mathematical viewpoint, in my opinion, as well as that of a number of colleagues.

mathematical contexts. Indeed, without sites, or more generally without presentations for toposes, we do not have the possibility of embodying in a topos a mathematical content stemming from a specific problem or situation (in a certain field of mathematics) that we would like to study, and therefore of applying topos-theoretic techniques to it. It is a bit like flying in an plane that is too high up to be able to see the ground and say anything interesting about it.

The refusal of the *essential ambiguity* resulting from the fact that a topos is associated with an infinity of different presentations has been justified by some members of this school as a refusal to work with toposes through their presentations rather than intrinsically (or axiomatically). This concern is entirely reasonable; however, the point here is not to study toposes through sites, but to study sites (or geometric theories or other objects apt to present toposes) through toposes!

The choices made by this school are, in fact, based on a bias against both infinitary and higher-order constructions. Now, Grothendieck's concept of topos is both infinitary (since, on the one hand, toposes are presented by *generally infinite sets* of generators, which is why they do not admit a finitary axiomatisation in the language of categories, and, on the other hand, since they are naturally tailored for infinitary operations such as arbitrary colimits) and of higher order (due to the fact that the concept of site is second order).

One can find many illustrations of this bias in the mathematical works of this school. For example, [9] proves several 'site characterisations' for invariant properties of toposes that have the form 'a topos satisfies the invariant if and only if *there exists* a site of definition for this topos with such or such property' (as if the site itself were unimportant), whereas, as we shall see in the next section, what one needs for constructing 'bridges' are characterisations of the form 'a topos $\mathbf{Sh}(\mathscr{C}, J)$ satisfies the invariant if and only if the site $(\mathscr{C}, J)$ satisfies such or such property', as one needs to move both from sites to toposes and vice versa (in other words, to 'ascend' from one site to the topos via one of the arches of the bridge and 'descend' on the other side through the other arch).

The rejection of higher-order constructions in this school also manifested in their attempts to develop a theory of relative toposes by using the (very rigid) concept of a category internal to a topos, whereas the correct concept (both from a geometric point of view and from the point of view of higher categories) is that of *stack* over a topos, as Grothendieck and Giraud had already understood: it suffices to think that the canonical stack of a topos, which is the central concept around which all the relative theory should be developed, *is not* an internal category![14] They also neglected the notion of *fibred site* introduced in [1] (see Lecture VI therein), which, when suitably generalised, gives rise to a much broader and flexible theory than the formalisms based on their notion of internal site.

[14] It is true that any stack is equivalent (but not necessarily isomorphic) to a split stack, and hence to an internal category, but this process of rigidification is based on choices (to define one of the functors yielding the equivalence) and is not convenient in practice.

Surprisingly, some of the promoters of this school have even referred to Grothendieck to justify their bias, interpreting the following passages by Grothendieck (found respectively in the preface to the second edition of SGA 4 and in the introduction to Lecture IV of that work) as a recognition of the inessential character of sites (in the book [10], which presents these developments, there is even a section entitled “The topos is more important than the site”!):

> Our guiding principle has been to develop a language and notations that are already those which effectively serve for the different applications, so as not to lose touch with the ‘geometric’ (or ‘topological’) content of the various functors that one has to consider between sites. For this, the notions of topos and morphism of toposes seem to be the indispensable leading thread, and it is appropriate to give them a central place, the notion of site becoming an auxiliary technical notion.

> On the other hand, experience has taught that various situations in Mathematics should be considered mainly as a technical means of constructing the corresponding categories of sheaves (of sets), i.e. the corresponding ‘toposes’. It appears that all the really important notions related to a site [. . .] are in fact expressible directly in terms of the associated topos.. . .

It is true that Grothendieck never spoke of ‘bridges’ or dualities between the level of sites and that of toposes. However, in his school it was not even conceivable to study toposes independently of their presentations (i.e. of the objects from which they were constructed), given that the geometry of the situations under consideration was always embodied by sites associated with schemes.

In this respect, the introduction of the book *Topos Theory* by P. Johnstone, whose main focus and inspiration is the theory of elementary toposes, is particularly enlightening: in it, Johnstone notably talks about the “fundamental uselessness” of the existence theorem for classifying toposes (!), complains that “the full import of the maxim “the topos is more important than the site” seems never to have been appreciated by the Grothendieck school” and concludes that, unlike Grothendieck, he does not “view topos theory as a machine for the demolition of unsolved problems in algebraic geometry or anywhere else”.

At the same time, Grothendieck blames geometers both for having abandoned toposes and for having made them a badly considered subject in the mathematical community:

> For nearly fifteen years now, it has been part of the good taste in the ‘grand world’, to look down on anyone who dares to pronounce the word ‘topos’, unless it is in jest or they have the excuse of being a logician. (These are people known for being like no others and to whom you have to forgive certain fads. . .).

In this regard, I still remember the recommendation I received a few years ago, when I was still a post-doc, from a well-known algebraic geometer, a former student of Grothendieck, to remove the word ‘topos’ from all my papers and replace it with the expression, which he felt was more acceptable in his community, ‘category of sheaves on a site’! Needless to say, I did not follow that advice; it would have been absurd given the central role played in my work by the *invariant* nature of the concept of topos (in relation to different presentation sites).

Remarkably, what has been missing in both schools is an integration between the ‘concrete’ level of sites and the ‘abstract’ or ‘metamathematical’ level of toposes, an integration which, as we shall see in more detail in the next section, is the essential condition for the fruitful use of toposes as unifying spaces in mathematics. This entails working *at two levels*, which must neither be confused nor separated from each other. Indeed, these two levels play fundamentally different roles: that of toposes is the level of *unity*, where the invariants live, whereas that of sites (or more generally of presentations of toposes) is the level of *diversity*, where the variability of forms in which a given invariant manifests itself unfolds.

4 Toposes as ‘Bridges’: the Underlying Vision and Some Examples

Since my doctoral studies, I have been working to develop a theory and a number of techniques for using toposes as unifying ‘bridges’ between different mathematical theories or contexts. Remember that Grothendieck uses the word ‘unification’ to mean that the same type of object—a topos—can be associated with *a priori* very different mathematical situations.

Grothendieck does not speak of ‘*bridges*’ or ‘*transfers*’ of knowledge between different theories that would be made possible by toposes. However, this new perspective of toposes as ‘bridges’ seems to us to constitute a natural prolongation of Grothendieck’s metamathematical conception of toposes.

The theory of topos-theoretic ‘bridges’, introduced in the programmatic text [2], makes it possible to exploit the technical flexibility inherent to the notion of topos—more precisely, the possibility of representing toposes in a multitude of different ways—to construct ‘bridges’ unifying different mathematical theories with equivalent or closely related semantic contents.

In recent years, as well as leading to the resolution of long-standing problems in categorical logic, these general techniques have given rise to a number of non-trivial applications in different areas of mathematics, and the potential of this theory has only just begun to be exploited.

In fact, these ‘bridges’ are useful not only for *connecting* different mathematical theories with each other, but also for *studying* a given theory within a specific domain.

To illustrate the potential field of application of this theory, we provide a non-exhaustive list of mathematical fields in which it has produced substantial results:

- *Model theory* (interpretation and topos-theoretic generalisation of Fraïssé’s theorem)
- *Proof theory* (new deductive systems for geometric theories)
- *Algebra* (topos-theoretic generalisation of the Galois formalism)
- *Topology* (reinterpretation and generation of Stone-type and Priestley-type dualities)

– *Functional analysis* (results about Gelfand spectra and Wallman compactifications)
– *Lattice-ordered groups and MV-algebras* (papers with A. C. Russo)
– *Cyclic structures* introduced by A. Connes and C. Consani (work on 'cyclic theories' with N. Wentzlaff)
– *Algebraic geometry* (generalisation of Nori motives, with L. Bar-bieri-Viale and L. Lafforgue, and logical approach to ℓ-independence problems for ℓ-adic cohomology)

For a conceptual presentation of these and other results, the reader is referred to [7]. The general principles of the theory of toposes as 'bridges' are also presented in Chapter 2 of [6].

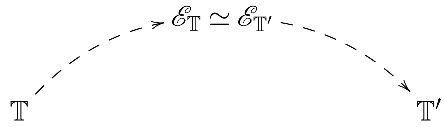
4.1 The Key Principles

The theory of toposes as 'bridges' is based on a meta-mathematical view of toposes, the key principles of which can be summarised as follows:

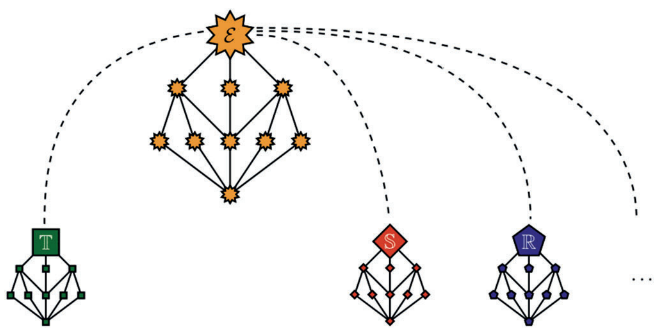
– The notion of Morita equivalence is ubiquitous in mathematics; in fact, in many situations it formalises the feeling of 'looking at the same thing in different ways' or 'constructing the same mathematical object by different methods'.
– In fact, several important dualities and equivalences in mathematics can be interpreted naturally in terms of Morita equivalences (see, e.g. my work on Stone-type dualities and on MV-algebras).
– On the other hand, topos theory itself is a primary source of Morita equivalences. Different representations of the same topos can be interpreted as Morita equivalences between different theories.
– Two *bi-interpretable* theories (i.e. whose coherent or geometric syntactic categories are equivalent) are Morita-equivalent, but, very remarkably, the converse is not true (see, e.g. our work on MV-algebras). In fact, most of Morita equivalences do not boil down to bi-interpretations or even to interpretations of one theory into the other; one must enrich these syntactic categories with *imaginaries* in the sense of model theory (which amounts to constructing their classifying toposes or pretoposes), in order to arrive at an equivalence of categories. This means that most of the correspondences (that can be established) in mathematics are invisible from a concrete point of view, as they are not induced by 'dictionaries'.
– Moreover, the notion of Morita equivalence captures the intrinsic *dynamism* inherent to the notion of mathematical theory; indeed, a mathematical theory gives rise *by itself* to an infinite number of Morita equivalences, thanks to the different points of view that one can have on it. For instance, each way of representing a theory as a quotient (i.e. a geometric extension in the same language) of a geometric theory $\mathbb{T}$ gives rise to a representation of its classifying

topos in terms of that of $\mathbb{T}$ (see the duality theorem between quotients and subtoposes in Chapter 2 of [6]).

- The existence of different theories having the same classifying topos translates, at the technical level, into the existence of different presentations (in particular, different sites of definition) of that topos.
- Topos-theoretic *invariants* can thus be used for transferring information from one theory to the other:



- *Transfers of information* are obtained by expressing a given invariant in terms of different representations for the topos. Every invariant behaves in this context like a 'pair of special glasses' which allows one to reveal information which is hidden in the given Morita equivalence. Different invariants allow to transfer different pieces of information.
- Thus, different properties (resp. constructions) considered in the context of theories classified by the same topos appear as different *manifestations* of a *unique* property (resp. construction) defined at the topos level.
- The *level of generality* of topos-theoretic invariants is ideal for capturing several important aspects of mathematical theories. Indeed, invariants of the classifying topos $\mathscr{E}_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ translate into interesting logical (syntactic or semantic) properties of $\mathbb{T}$.
- The fact that topos-theoretic invariants of topos often specialise into important properties or constructions of natural mathematical interest is a clear indication of the *centrality* of these concepts in mathematics. In fact, everything that happens at the level of toposes has 'uniform' ramifications throughout mathematics. For example, the following figure represents the lattice structure on the collection of subtoposes of a topos $\mathscr{E}$ which induces lattice structures on the collections of 'quotients' of geometric theories $\mathbb{T}$, $\mathbb{S}$, $\mathbb{R}$ classified by $\mathscr{E}$:



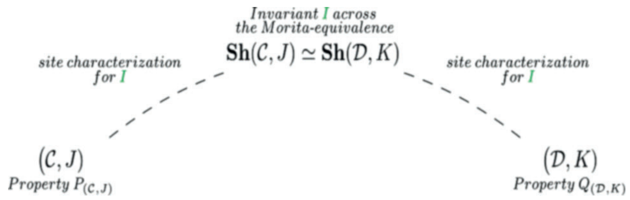
One of the reasons for the effectiveness of the topos-theoretic ‘bridge’ technique is that the relationship between a topos and its different presentations is *very natural*, which allows invariants to be (often easily but not trivially) transferred between different presentations (and hence between different theories). In fact, the degree of complexity of these ‘translations’ varies enormously from one invariant to another: there are large classes of invariants for which characterisations, in terms of sites or other topos presentations, can be established in an essentially automatic way, whereas for other invariants, notably the cohomological ones, the computations can be very difficult even in special cases. At the same time, this method is liable to produce profound and surprising results, since a given invariant can manifest itself in very different ways in the context of different presentations. In fact, this is a kind of *mathematical morphogenesis*, resulting from the expression of invariants in terms of different presentations of toposes. By considering different invariants in the context of the same equivalence of toposes, the ‘bridge’ method enables one to generate a wealth of results around a given theme (embodied by that equivalence). This way of doing mathematics is therefore characterised by a very high degree of *continuity* and *modularity*. Indeed, on the one hand, the generality of the method makes it possible to *adapt* or *transport* concepts, techniques and results from one context to another, while, on the other hand, the study of topos-theoretic invariants allows us to identify, in concrete mathematical contexts, the ‘good notions’, i.e. the concepts that correspond to topos-theoretic invariants (via their characterisations in terms of sites or other presentations of toposes) and therefore admit an infinite number of equivalent reformulations in other mathematical contexts.

A topos-theoretic ‘bridge’ is a structure which involves two levels, the ‘concrete’ one of sites or, more generally, of objects liable of presenting toposes and the ‘abstract’ one of toposes themselves; the characterisations of topos-theoretic invariants in terms of topos presentations form the ‘arches’ of ‘bridges’. Schematically, a typical ‘bridge’ has the following form:

- *Decks* of ‘bridges’: *Morita equivalences* (or, more generally, morphisms of toposes or other kinds of relations between them)
- *Arches* of ‘bridges’: *Characterisations in terms of sites* (or, more generally, expressions of topos-theoretic invariants in terms of different topos presentations)

The different expressions of a given invariant in the context of different presentations of the same topos thus find themselves logically linked by the ‘bridge’. In this way, one manages to relate different concrete properties by means of topos invariants. Note that toposes (and their invariants) no longer appear in the final formulation of such a logical relation, despite having played a crucial role in its discovery. This serves as an illustration of the need to make a ‘leap’ into the ‘imaginary’ (setting in which symmetries naturally manifest themselves, or where the invariants ‘live’) in order to be able to link together ‘real’ entities, as well as to gain a more global understanding of the phenomenon we wish to study and to enjoy greater computational possibilities.

For example, in the following ‘bridge’, we have an invariant property I and logical equivalences ‘$\mathbf{Sh}(\mathscr{C}, J)$ satisfies I if and only if the site $(\mathscr{C}, J)$ satisfies the property $P_{(\mathscr{C},J)}$' and '$\mathbf{Sh}(\mathscr{D}, K)$ satisfy I if and only if the site $(\mathscr{D}, K)$ satisfies the property $Q_{(\mathscr{D},K)}$', which constitute the arches of the 'bridge' and which allow us to establish the equivalence between $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$. Note that these two properties are unified by this bridge as they are interpreted as *manifestations* of a *unique* property I defined at the topos level:



4.2 Some Examples of 'Bridges'

To illustrate the applicability of the technique of toposes as 'bridges', let us briefly discuss a few examples:

- Theories of presheaf type
- Topos-theoretic Fraïssé theorem
- Topological Galois theory
- Stone-type dualities

In fact, the results obtained in each of these subjects are completely *different*, but the underlying methodology is always the *same*!

Theories of Presheaf Type

Recall that a geometric theory is said to be *of presheaf type* if it is classified by a presheaf topos.

Theories of presheaf type are very important because they constitute the 'building blocks' from which any geometric theory can be constructed. In fact, just as any Grothendieck topos can be written as a subtopos of some presheaf topos, any geometric theory can be written as a 'quotient' of some theory of presheaf type.

All finitary algebraic (or, more generally, cartesian) theories are of presheaf type, but this class of theories contains many other interesting mathematical theories (see, e.g. Chapter 9 of [6]).

Every theory of presheaf type $\mathbb{T}$ possesses two different natural representations of its classifying topos, which can be used to construct 'bridges' linking its *syntax* and its *semantics*:

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \qquad\qquad\qquad\qquad (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Here, f. p .$\mathbb{T}$-mod(**Set**) denotes the category of finitely presentable models of $\mathbb{T}$ and $(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the geometric syntactic site of $\mathbb{T}$.

Let us consider, for instance, in the context of a theory of presheaf type $\mathbb{T}$, which we assume for the sake of simplicity to have only one sort, the topos-theoretic invariant given by the notion of subobject of the product $U \times \cdots \times U$ (n times) of the universal model U of $\mathbb{T}$. This invariant expresses itself, in terms of the representation $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of the classifying topos of $\mathbb{T}$, as the notion of geometric formula in n variables, considered up to $\mathbb{T}$-provable equivalence, and, in terms of the representation [f. p .$\mathbb{T}$-mod(**Set**), **Set**], as the notion of functorial property of n-uplets of elements of finitely presentable models M of $\mathbb{T}$:

Subobject of $U \times \cdots \times U$

$$[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \qquad\qquad\qquad\qquad (\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

Functorial correspondence $M \to R_M \subseteq M \times \cdots \times M$ — Geometric formula $\phi(x_1, \ldots, x_n)$

This 'bridge' thus leads to the following definability theorem:

Theorem *Let $\mathbb{T}$ be a theory of presheaf type. Let us suppose that we are given, for each finitely presentable set-based model $\mathscr{M}$ of $\mathbb{T}$, a subset $R_{\mathscr{M}}$ of $\mathscr{M}^n$ in such a way that every homomorphism $h : \mathscr{M} \to \mathscr{N}$ of finitely presentable $\mathbb{T}$-models sends $R_{\mathscr{M}}$ into $R_{\mathscr{N}}$. Then there exists a geometric formula $\phi(x_1, \ldots, x_n)$ which, for any finitely presentable $\mathbb{T}$-model $\mathscr{M}$, defines the subset $R_{\mathscr{M}}$.*

Of course, the consideration of other invariants allows one to obtain many other interesting results about theories of presheaf type (for this, the reader is referred to [6]).

Topos-Theoretic Fraïssé Theorem

Let us now show that Fraïssé's theorem in model theory admits a substantial generalisation arising from a triple 'bridge'.

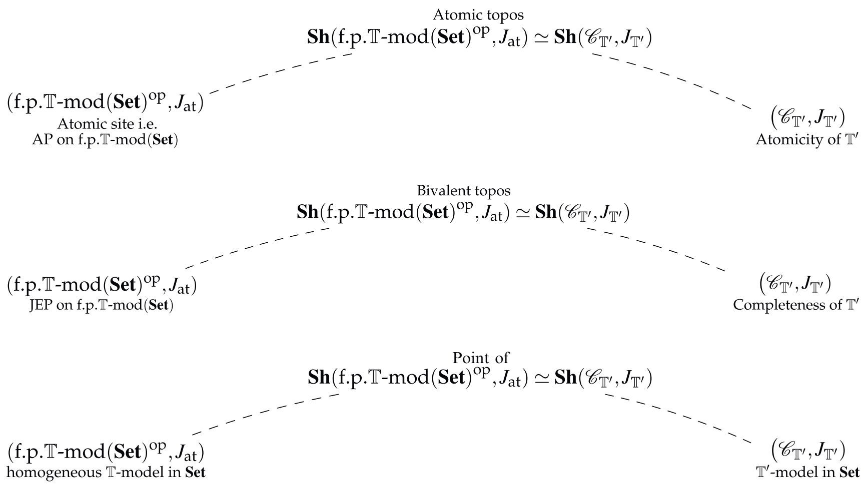
Let us first note that, in the context of presheaf type theories, we can introduce a notion of homogeneous model which generalises that of weakly homogeneous model in classical model theory:

Definition A set-based model M of a theory of a geometric theory $\mathbb{T}$ is said to be *homogeneous* if for any arrow $f : c \to d$ in f. p .$\mathbb{T}$-mod(**Set**) and any arrow $y : c \to M$ in $\mathbb{T}$-mod(**Set**), there exists an arrow $u : d \to M$ in $\mathbb{T}$-mod(**Set**) such that $u \circ f = y$:

$$\begin{array}{ccc} c & \xrightarrow{y} & M \\ {\scriptstyle f}\downarrow & \nearrow_{u} & \\ d & & \end{array}$$

We can also define, in the context of categories, the properties of amalgamation (AP) and joint embedding (JEP): we will say that a category satisfies AP if any pair of arrows with common domain can be completed into a commutative square, and that it satisfies JEP if any two objects admit an arrow to a third object.

Now, the invariants defined by the properties of a topos to be *bivalent* and to be *atomic*, and the invariant notion of *point* of a topos each give rise to a different 'bridge' whose deck is the same Morita equivalence:



Putting these three 'bridges' together, we obtain the following theorem (from [3]), which constitutes a broad generalisation of Fraïssé's classical theorem (which is the particular case where $\mathbb{T}$ is the quotient of the theory over a finite language corresponding to a uniformly finite collection of finitely presentable models of this theory which satisfies the hereditary property):

Theorem *Let* $\mathbb{T}$ *be a theory of presheaf type such that the category* f.p.$\mathbb{T}$-mod(**Set**) *is non-empty and satisfies AP. Then the theory* $\mathbb{T}'$ *of homogeneous* $\mathbb{T}$*-models is atomic, and it is complete if and only if* f.p.$\mathbb{T}$-mod(**Set**) *satisfies JEP.*

The level of generality of this topos-theoretic interpretation of Fraïssé's theorem is high enough to allow for a unification of Fraïssé's theory with Galois theory, as we explain in the next section.

It is interesting to note that the consideration of these *three* invariants in the context of equivalence

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$$

has allowed us to recover and improve Fraïssé's theorem; still, there is an infinite number of topos-theoretic invariants which we can consider in in relation to this same equivalence! In this way, one can generate other results, related to Fraïssé's theorem but in general independent of it as well as independent of each other (examples of such results are given in [3]). This is an illustration of how the 'bridge' method naturally generates *families of results* around a given theme.

Topological Galois Theory

Before presenting our topos-theoretic interpretation of Galois theory, we need to introduce the following categorical notions:

Definition Let $\mathbb{T}$ be a theory of presheaf type. A set-based model M of $\mathbb{T}$ is said to be $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$-universal if every finitely presentable model of $\mathbb{T}$ admits a homomorphism towards M, and it is said to be $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$-ultrahomogeneous if all the arrows from a finitely presentable model of $\mathbb{T}$ towards M can be transformed one into another by composition with an automorphism of M.

The following representation theorem (from [4]) provides the deck of 'bridges' which we will use to generate concrete Galois theories:

Theorem *Let $\mathbb{T}$ be a theory of presheaf type such that $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ satisfies AP and JEP, and let M be a $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$-universal and $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$-ultrahomogeneous model of $\mathbb{T}$. Then we have an equivalence of toposes*

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Cont}(\text{Aut}(M)),$$

where the automorphism group $\text{Aut}(M)$ of M is endowed with the topology of pointwise convergence. This equivalence is induced by the functor

$$F : \text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}} \to \mathbf{Cont}(\text{Aut}(M))$$

which sends any model c of $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ to the set $\text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(c, M)$ (endowed with the obvious action of $\text{Aut}(M)$) and every arrow $f : c \to d$ in $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ to the $\text{Aut}(M)$-equivariant map

$$- \circ f : \text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(d, M) \to \text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(c, M) .$$

Note that the functor F takes its values in the full subcategory $\mathbf{Cont}_t(\text{Aut}(M))$ of $\mathbf{Cont}_t(\text{Aut}(M))$ on the transitive and non-empty actions (which is the category of atoms of the topos $\mathbf{Cont}_t(\text{Aut}(M))$), but it is not necessarily an equivalence towards this category, nor is it necessarily full and faithful.

The following result gives necessary and sufficient conditions for F to be full and faithful (resp. an equivalence towards $\mathbf{Cont}_t(\text{Aut}(M))$).

Theorem *Under the assumptions of the last theorem, the functor* F *is fully faithful if and only if every arrow of* $\mathrm{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ *is a strict monomorphism, and it is an equivalence onto the subcategory* $\mathbf{Cont}_t(\mathrm{Aut}(M))$ *of* $\mathbf{Cont}(\mathrm{Aut}(M))$ *if and only if moreover* $\mathrm{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ *is atomically complete.*

This result springs from two 'bridges', which are obtained by considering the invariant notions of *atom* and of *arrow between atoms*:

$$\mathbf{Sh}(\mathrm{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\mathrm{op}}, J_{\mathrm{at}}) \simeq \mathbf{Cont}(\mathrm{Aut}(M))$$

$$\mathrm{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\mathrm{op}} \qquad\qquad \mathbf{Cont}_t(\mathrm{Aut}(M))$$

Indeed, a category $\mathscr{C}$ whose opposite satisfies AP is said to be atomically complete if all the atoms of the topos $\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{\mathrm{at}})$ are isomorphic to an object of the form $l(c)$ for an object c of $\mathscr{C}$ (where l is the canonical functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{\mathrm{at}})$). This condition admits an elementary characterisation in the language of categories.

This theorem generalises Grothendieck's theory of Galois categories and can be applied to obtain Galois-type theories in different domains of mathematics, for example, that of finite groups and that of finite graphs.

The reader will perhaps have noticed that the toposes involved in Fraïssé's theory are precisely the atomic and bivalent ones, which are also the toposes involved in our interpretation of Galois theory. In fact, Galois theory consists in studying these topos from the point of view of group theory, by representing them as categories of continuous actions of topological groups on discrete sets, whereas Fraïssé's theory consists in studying them from a syntactic point of view. For more details on these results and the unification between Fraissé's theory and Galois theory, we refer the reader to [4].

Stone-Type Dualities

In [5] we show that all the Stone-type dualities or equivalences between particular kinds of preorders, locales or topological spaces can be obtained by *functorialising* 'bridges' of the form

$$\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \simeq \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$$

$$\mathscr{C} \qquad\qquad \mathscr{D}$$

where $\mathscr{D}$ is a $J_{\mathscr{C}}$-dense subcategory of a category $\mathscr{C}$ defined by a preorder. These equivalences are notably provided by Grothendieck's *Comparison Lemma*. For instance, we can take for $\mathscr{D}$ a Boolean algebra and for $\mathscr{C}$ the lattice of open sets of its Stone space to obtain *Stone duality*, or for $\mathscr{C}$ a complete atomic Boolean algebra and for $\mathscr{D}$ the collection of its atoms to obtain *Lindenbaum-Tarski duality*.

This method makes it possible to generate many new dualities for other kinds of preordered structures (e.g. a localic duality for meet-semilattices, a duality for k-frames, a duality for disjunctively distributive lattices, a duality for preframes generated by their directly irreducible elements, etc.). It also generalises naturally to arbitrary categories.

This work notably illustrates the approach to dualities via the theory of 'bridges': to relate two objects, rather than trying to find categories into which they can be inserted and then seek to construct adjunctions or equivalences between them (e.g. by using Hom functors to a 'dualising' object, as in the classical categorical approach), one concentrates on the two objects and tries to embody their common invariants in a single topos associated with each of them. We thus obtain a 'bridge' for each pair of objects, and the consideration of different morphisms between the associated toposes provides different ways of 'functorialising' these 'bridges' and thus obtaining equivalences between categories containing these two objects. Note, however, that the identification of these categories is not given a priori (as in the classical categorical approach); it *results* from the 'bridges' induced by invariant notions of topos morphisms.

On the other hand, the classical approach of the category theorists is based on a principle of continuity which, although intuitively plausible, runs the risk of drowning out the specificity of the objects in question in the abstraction of the categories in which one decides to consider them, whereas associating each object with a topos embodying its 'essence' (to use the Grothendieckian terminology) is liable to valorise the diversity of the individual objects in a much more profound way. It's a bit like demanding that, in order for two individuals be able to marry, the families or social groups to which they belong should 'marry' as well; some people might find this desirable, but there is little doubt that this is an overly restrictive condition!

Another interesting aspect of this approach is that it produces a veritable *machine for generating dualities* which, on the one hand, makes it possible to recover the various classical dualities and, on the other hand, to establish many new ones. Remarkably, thanks to this unifying point of view, we can figure out why certain dualities had been discovered while others had remained hidden, even though they have the same level of mathematical 'depth', being generated by the same 'machine': often, the reason for this lies in the linguistic complexity of their description, which can vary significantly from one duality to another, so that the human intuition may suffice to discover the simplest ones (but not the others) in the absence of a general theory. Several illustrations of this remark are given in [5].

5 Future Perspectives

The results obtained so far show that toposes can play an effective role as *unifying spaces* for transferring information between different mathematical theories and generating new equivalences, dualities and correspondences across various areas of mathematics.

In fact, toposes have a real creative power in mathematics, in the sense that their study naturally generates, in the most diverse mathematical contexts, a large number of relevant but often unsuspected 'concrete' notions and results.

In the coming years, we intend to pursue both theoretical and applied research to further develop the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as 'key' concepts defining a new way of doing mathematics that can shed unique light on a wide range of different topics.

Central themes in our research programme will be:

- Studying important dualities or correspondences in mathematics from a topos-theoretic point of view (in particular, the theory of motives, class field theory and the Langlands programme)
- Systematically investigating expressions of topos-theoretic invariants in terms of different presentations of toposes and defining new invariants that capture important aspects of concrete mathematical problems
- Interpreting and generalising important parts of model theory (both classical and modern) in terms of toposes and developing a functorial model theory
- Introducing new methodologies for generating Morita equivalences
- Developing general techniques for building *spectra* by using classifying toposes
- Generalising the 'bridge' technique to the context of higher categories and toposes by developing a higher-order geometric logic
- Developing a theory of classifying topos over arbitrary base toposes and *relativisation techniques* for concepts and results

With the development of this programme, the topos-theoretic study of mathematical theories should become increasingly 'user-friendly' and hence easily applicable not only to mathematics but also to the investigation of fundamental problems in other sciences, especially theoretical physics and computer science. It would be interesting, for instance, to study whether relativity theory and quantum mechanics could be unified through the identification of common toposes associated with one theory and the other (which would thus be constructed from sites of analytical nature in the first case and objects arising from the theory of operator algebras or non-commutative geometry, such as C^*-algebras or quantales/quantaloides, in the second case). Another natural subject of study for a possible topos-theoretic interpretation would be that of important dualities in physics such as the AdS/CFT correspondence and mirror symmetry.

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